

RATIONAL SOLUTIONS OF THE SCHLESINGER SYSTEM AND ISOPRINCIPAL DEFORMATIONS OF RATIONAL MATRIX FUNCTIONS II

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ABSTRACT. In this second article of the series we study holomorphic families of generic rational matrix functions parameterized by the pole and zero loci. In particular, the isoprincipal deformations of generic rational matrix functions are proved to be isosemiresidual. The corresponding rational solutions of the Schlesinger system are constructed and the explicit expression for the related tau function is given. The main tool is the theory of joint system representations for rational matrix functions with prescribed pole and zero structures.

NOTATION

- \mathbb{C} stands for the complex plane.
- \mathbb{C}_* stands for the punctured complex plane:

$$\mathbb{C}_* = \mathbb{C} \setminus \{0\}.$$

- $\overline{\mathbb{C}}$ stands for the extended complex plane (= the Riemann sphere):

$$\overline{\mathbb{C}} = \mathbb{C} \cup \infty.$$

- z stands for the complex variable.
- \mathbb{C}^n stands for the n -dimensional complex space.
- In the coordinate notation, a point $\mathbf{t} \in \mathbb{C}^n$ will be written as $\mathbf{t} = (t_1, \dots, t_n)$.
- \mathbb{C}_*^n is the set of points $\mathbf{t} \in \mathbb{C}^n$, whose coordinates t_1, \dots, t_n are pairwise different:

$$\mathbb{C}_*^n = \mathbb{C}^n \setminus \bigcup_{\substack{1 \leq i, j \leq n \\ i \neq j}} \{\mathbf{t} : t_i = t_j\}.$$

- $\mathbb{C}^{m \times n}$ stands for the set of all $m \times n$ matrices with complex entries.
- For $A \in \mathbb{C}^{m \times n}$, $A^* \in \mathbb{C}^{n \times m}$ is the adjoint matrix, $\text{Im}(A)$ is the image subspace of A in \mathbb{C}^m (= the linear span of the columns of A) and $\text{Nul}(A)$ is the null subspace of A in \mathbb{C}^n .
- $[\cdot, \cdot]$ denotes the commutator: for $A, B \in \mathbb{C}^{m \times m}$, $[A, B] = AB - BA$.
- I stands for the identity matrix of an appropriate dimension.

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10. SIMPLE SINGULARITY OF A MEROMORPHIC MATRIX FUNCTION

(For Sections 1-9 see [KaVo], [KaVo1-e].)

Definition 10.1. Let $R(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic in a punctured neighborhood of a point $t \in \mathbb{C}$. The point t is said to be a *simple pole* of the matrix function R if

$$R(z) = \frac{R_t}{z-t} + H(z),$$

where $R_t \in \mathbb{C}^{m \times m}$ is a constant matrix and the function H is holomorphic at the point t . The matrix R_t is said to be the *residue* of the function R at the point t . Furthermore, if $r = \text{rank}(R_t)$ and $f_t \in \mathbb{C}^{m \times r}$ and $g_t \in \mathbb{C}^{r \times m}$ are matrices providing the factorization $R_t = f_t g_t$, we shall say that f_t is the *left semiresidue* of R at t and g_t is the *right semiresidue* of R at t .

Remark 10.2. The left and right semiresidues f_t, g_t are defined up to the transformation

$$f_t \rightarrow f_t c, \quad g_t \rightarrow c^{-1} g_t,$$

where $c \in \mathbb{C}^{r \times r}$ is an invertible matrix.

Definition 10.3. Let $R(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in a punctured neighborhood of a point $t \in \mathbb{C}$.

- (1) The point t is said to be *regular* for the function R if both the function R and the inverse function R^{-1} are holomorphic functions in the entire (non-punctured) neighborhood of the point t , i.e. if R and R^{-1} are holomorphic at the point t .
- (2) The point t is said to be *singular* for the function R if at least one of the functions R and R^{-1} is not holomorphic at the point t .

In particular, the point t is singular for the function R if R is holomorphic at the point t , but its value $R(t)$ is a degenerate matrix. In this case, the point t is said to be a *zero* of the function R .

Definition 10.4. Let $R(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in a punctured neighborhood of a point $t \in \mathbb{C}$, and let t be a singular point of R . The singular point t is said to be *simple* if one of the following holds:

- (1) The point t is a simple pole of the function R and a holomorphy point of the inverse function R^{-1} .
- (2) The point t is a simple pole of the inverse function R^{-1} and a holomorphy point of the function R itself.

Remark 10.5. Note that, according to Definition 10.4, if t is a simple singular point of the function R then R is a single-valued meromorphic function in the entire (non-punctured) neighborhood of t .

Our main goal is to study a matrix function in a neighborhood of its simple singular point from the point of view of linear differential systems. Thus we consider the left logarithmic derivative of the function R :

$$Q_R^l(z) \stackrel{\text{def}}{=} R'(z)R(z)^{-1}.$$

Remark 10.6. One can also consider the right logarithmic derivative of R :

$$Q_R^r(z) = R(z)^{-1}R'(z).$$

But then $-Q_R^r$ is the left logarithmic derivative of the inverse function R^{-1} :

$$Q_{R^{-1}}^l(z) = (R^{-1}(z))'R(z) = -R(z)^{-1}R'(z)R(z)^{-1}R(z) = -Q_R^r(z).$$

Thus in this work we shall deal mainly with left logarithmic derivatives. Therefore, we shall use the notation Q_R instead of Q_R^l :

$$Q_R(z) \stackrel{\text{def}}{=} R'(z)R(z)^{-1},$$

and omit the word "left" when referring to the left logarithmic derivative.

Proposition 10.7. *Let $R(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in a punctured neighborhood of a point $t \in \mathbb{C}$, and let t be a simple singular point of R . Then the point t is a simple pole for the logarithmic derivative¹ Q_R of R . Moreover, for the residue and the constant term of the Laurent expansion*

$$(10.1) \quad Q_R(z) = \frac{Q_t}{z-t} + C + o(1) \text{ as } z \rightarrow t$$

the following relations hold.

(1) *If t is a pole of R then*

$$(10.2a) \quad Q_t^2 = -Q_t,$$

$$(10.2b) \quad Q_t C Q_t = -C Q_t$$

and

$$(10.3) \quad \text{Im}(Q_t) = \text{Im}(R_t),$$

where R_t is the residue of R at t .

(2) *If t is a zero of R then*

$$(10.4a) \quad Q_t^2 = Q_t,$$

$$(10.4b) \quad Q_t C Q_t = Q_t C$$

and

$$(10.5) \quad \text{Nul}(Q_t) = \text{Nul}(R_t),$$

where R_t is the residue of R^{-1} at t .

Proof. First, let us assume that t is a pole of R and let

$$(10.6) \quad R(z) = \frac{R_t}{z-t} + A_0 + A_1(z-t) + A_2(z-t)^2 + \dots,$$

$$(10.7) \quad R^{-1}(z) = B_0 + B_1(z-t) + B_2(z-t)^2 + \dots,$$

be the Laurent expansions of the functions R and R^{-1} at t . Then

$$(10.8) \quad R'(z) = -\frac{R_t}{(z-t)^2} + A_1 + 2A_2(z-t) + \dots$$

Multiplying the Laurent expansions term by term, we obtain from (10.7) and (10.8)

$$(10.9) \quad Q_R(z) = -\frac{R_t B_0}{(z-t)^2} - \frac{R_t B_1}{z-t} - R_t B_2 + A_1 B_0 + o(1).$$

¹See Remark 10.6.

Substituting the expansions (10.6), (10.7) into the identity

$$R^{-1}(z)R(z) = R(z)R^{-1}(z) = I,$$

we observe that

$$(10.10) \quad R_t B_0 = B_0 R_t = 0 \text{ and } R_t B_1 + A_0 B_0 = I.$$

Hence the first term of the expansion (10.9) vanishes and we obtain the expansion (10.1) with

$$(10.11) \quad Q_t = -R_t B_1 = A_0 B_0 - I,$$

$$(10.12) \quad C = -R_t B_2 + A_1 B_0.$$

Thus

$$(I + Q_t)Q_t = (A_0 B_0)(-R_t B_1) = -A_0(B_0 R_t)B_1 = 0,$$

i.e. (10.2a) holds. Furthermore,

$$\begin{aligned} (I + Q_t)CQ_t &= (A_0 B_0)(-R_t B_2 + A_1 B_0)(-R_t B_1) = \\ &= A_0(B_0 R_t)B_2 R_t B_1 - A_0 B_0 A_1 (B_0 R_t)B_1 = 0, \end{aligned}$$

i.e. (10.2b) holds as well. Finally,

$$Q_t R_t = (A_0 B_0 - I)R_t = A_0(B_0 R_t) - R_t = -R_t,$$

which, together with (10.11), implies (10.3). This completes the proof in the case when t is a pole of R . The case when t is a zero of R can be treated analogously. \square

Remark 10.8. Since for any $p \times q$ matrix A the subspace $\text{Nul}(A)$ is the orthogonal complement of the subspace $\text{Im}(A^*)$ in \mathbb{C}^q , the relation (10.5) can be rewritten as

$$\text{Im}(Q_t^*) = \text{Im}(R_t^*).$$

The latter relation, together with (10.4a), means that Q_t^* is a (non-orthogonal, in general) projector onto the subspace $\text{Im}(R_t^*) \subset \mathbb{C}^m$. Hence the right semiresidue of R^{-1} at its pole t is also the right semiresidue of Q_R at t .

Analogously, the relations (10.2a) and (10.3) mean that $-Q_t$ is a (non-orthogonal, in general) projector onto the subspace $\text{Im}(R_t) \subset \mathbb{C}^m$. Hence the left semiresidue of R at its pole t is also the left semiresidue of Q_R at t .

Proposition 10.7 implies that a $\mathbb{C}^{m \times m}$ -valued function $R(z)$ in a punctured neighborhood of its simple singular point t may be viewed as a fundamental solution of a linear differential system

$$(10.13) \quad R'(z) = Q(z)R(z),$$

for which t is a *Fuchsian singularity* (see the first part of this work [KaVo] for details and references) and whose coefficients satisfy the relations (10.2) or (10.4). The next proposition shows that (10.2) or (10.4) are the only requirements a differential system (10.13) with a Fuchsian singularity t has to satisfy in order for its fundamental solution in a punctured neighborhood of t to be single-valued and have a simple singular point at t :

Proposition 10.9. *Let $Q(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and single-valued in a punctured neighborhood Ω of a point t . Let the point t be a simple pole for $Q(z)$, let*

$$(10.14) \quad Q(z) = \frac{Q_t}{z - t} + C + o(1) \text{ as } z \rightarrow t$$

be the Laurent expansion of the function Q at the point t and let R be a fundamental solution of the linear differential system

$$(10.15) \quad R'(z) = Q(z)R(z), \quad z \in \Omega.$$

Assume that one of the following two cases takes place.

(1) The coefficients Q_t, C of the expansion (10.14) satisfy the relations

$$(10.16a) \quad Q_t^2 = -Q_t,$$

$$(10.16b) \quad Q_t C Q_t = -C Q_t.$$

(2) The coefficients Q_t, C of the expansion (10.14) satisfy the relations

$$(10.17a) \quad Q_t^2 = Q_t,$$

$$(10.17b) \quad Q_t C Q_t = Q_t C.$$

Then R is a single-valued function in Ω and t is a simple singular point of R ; in the first case t is a pole of R , in the second case t is a zero of R .

Proof. Once again, we shall prove only the first statement. Thus we assume that the relations (10.2a), (10.2b) hold and consider the transformation

$$U(z) = (I + Q_t + (z - t)Q_t)R(z).$$

Then, because of (10.16a), the inverse transformation is given by

$$R(z) = (I + Q_t + (z - t)^{-1}Q_t)U(z).$$

Substituting these formulae and the Laurent expansion of M into the linear system (10.15), we obtain the following linear system for U :

$$U'(z) = \left(\frac{(I + Q_t)CQ_t}{z - t} + V(z) \right) U(z),$$

where the function $V(z)$ is holomorphic in the entire (non-punctured) neighborhood of the point t . In view of (10.16b), the coefficients of this system are holomorphic at the point t , hence U is holomorphic and invertible in the entire neighborhood of t and R has a simple pole at t . Since

$$R^{-1}(z) = U^{-1}(z)(I + Q_t + (z - t)Q_t),$$

R^{-1} is holomorphic at t and hence has a zero at t . \square

An important role in the theory of Fuchsian differential systems is played by multiplicative decompositions of fundamental solutions (see Section 5 of [KaVo]). In the present setting we are interested in decompositions of the following form:

Definition 10.10. Let $R(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in a punctured neighborhood Ω of a point t . Let R admit in Ω the factorization

$$(10.18) \quad R(z) = H_t(z)E_t(\zeta), \quad \zeta = z - t, \quad z \in \Omega,$$

where the factors $H_t(z)$ and $E_t(\zeta)$ possess the following properties:

- (1) $H_t(z)$ is a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in the entire neighborhood $\Omega \cup \{t\}$;
- (2) $E_t(\zeta)$ is a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in the punctured plane $\mathbb{C}_* = \mathbb{C} \setminus 0$.

Then the functions $E_t(\zeta)$ and $H_t(z)$ are said to be, respectively, the *principal* and *regular factors* of R at t .

Remark 10.11. The multiplicative decomposition (10.18), which appears in Definition 10.10, is always possible. This follows, for example, from the results due to G.D.Birkhoff (see [Birk1]). The principal factor $E_t(\zeta)$ is, in a sense, the multiplicative counterpart of the principal part of the additive (Laurent) decomposition: it contains the information about the nature of the singularity t of R . Of course, the principal and regular factors at the point t are determined only up to the transformation

$$(10.19) \quad E_t(\zeta) \rightarrow M(\zeta)E_t(\zeta), \quad H_t(z) \rightarrow H_t(z)M^{-1}(z-t),$$

where $M(z)$ is an invertible entire $\mathbb{C}^{m \times m}$ -valued function. However, once the choice of the principal factor E_t is fixed, the regular factor H_t is uniquely determined and vice-versa.

A possible choice of the principal factor of the function R at its simple singular point t is described in the following

Lemma 10.12. *Let $R(z)$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in a punctured neighborhood of a point t and let t be a simple singular point of R . Then a principal factor $E_t(\zeta)$ of R at t can be chosen as follows.*

(1) *If t is a pole of R , choose any matrix $L \in \mathbb{C}^{m \times m}$, satisfying the conditions*

$$(10.20) \quad L^2 = -L, \quad \text{Nul}(L) = \text{Nul}(R_t),$$

where R_t is the residue of R at t , and set for $\zeta \in \mathbb{C}_$*

$$(10.21) \quad E_t(\zeta) = I + L - \zeta^{-1}L.$$

(2) *If t is a zero of R , choose any matrix $L \in \mathbb{C}^{m \times m}$, satisfying the conditions*

$$(10.22) \quad L^2 = L, \quad \text{Im}(L) = \text{Im}(R_t),$$

where R_t is the residue of R^{-1} at t , and set for $\zeta \in \mathbb{C}_$*

$$(10.23) \quad E_t(\zeta) = I - L + \zeta L.$$

Proof. Let us assume that t is a pole of R and that the function E_t is given by (10.21), where the matrix L satisfies the conditions (10.20). Then $E_t(\zeta)$ is holomorphic in \mathbb{C}^* ; its inverse $E_t^{-1}(\zeta)$ is given by

$$E_t^{-1}(\zeta) = I + L - \zeta L$$

and is holomorphic in \mathbb{C}^* , as well. Let us now show that the function

$$H(z) \stackrel{\text{def}}{=} R(z)E_t^{-1}(z-t)$$

is holomorphic and invertible at t .

Indeed, in a neighborhood of t the principal part of the Laurent expansion of H equals to $\frac{R_t(I+L)}{z-t}$. But by (10.20) $\text{Im}(L^*) = \text{Im}(R_t^*)$ and hence

$$\text{Im}((I+L^*)R_t^*) = \text{Im}((I+L^*)L^*) = \text{Im}((L^2+L)^*) = \{0\}.$$

Therefore, $R_t(I+L) = 0$ and H is holomorphic at t .

In the same way, the principal part of the Laurent expansion of H^{-1} equals to $-\frac{LB_0}{z-t}$, where $B_0 = R^{-1}(t)$ is the constant term of the Laurent expansion of R^{-1} at t . But $R_t B_0 = 0$ (see (10.10) in the proof of Proposition 10.7), hence

$$\operatorname{Im}(B_0^* L^*) = \operatorname{Im}(B_0^* R_t^*) = \{0\},$$

$LB_0 = 0$ and H^{-1} is holomorphic at t , as well.

The proof in the case when t is a zero of R is completely analogous. \square

Remark 10.13. Let us note that the formulae (10.21) and (10.23) can be rewritten in the unified form

$$E_t(\zeta) = \zeta^L (= e^{L \log \zeta}).$$

This is precisely the form of the principal factor (with $\hat{Q} = 0$) which appears in Proposition 5.6 of [KaVo].

Remark 10.14. The relations (10.20) mean that $-L^*$ is a projector onto $\operatorname{Im}(R_t^*)$. This is equivalent to L being of the form $L = pg_t$, where g_t is the right semiresidue of the function R at its pole t and $p \in \mathbb{C}^{m \times \operatorname{rank}(R_t)}$ is such that $g_t p = -I$. Analogously, the relations (10.22) mean that L is a projector onto $\operatorname{Im}(R_t)$. This is equivalent to L being of the form $L = f_t q$, where f_t is the left semiresidue of the function R^{-1} at its pole t and $q \in \mathbb{C}^{\operatorname{rank}(R_t) \times m}$ is such that $q f_t = I$. For example, one can choose the matrix L mentioned in Lemma 10.12 as follows:

$$L = \begin{cases} -g_t^*(g_t g_t^*)^{-1} g_t & \text{if } t \text{ is a pole of } R, \\ f_t(f_t^* f_t)^{-1} f_t^* & \text{if } t \text{ is a zero of } R. \end{cases}$$

11. RATIONAL MATRIX FUNCTIONS OF SIMPLE STRUCTURE

In this section we apply the local results obtained in Section 10 to the study of rational matrix functions.

Definition 11.1. A $\mathbb{C}^{m \times m}$ -valued rational function $R(z)$ is said to be a rational matrix function of *simple structure* if it meets the following conditions:

- (1) $\det R(z) \neq 0$;
- (2) all singular points of R are simple;
- (3) $z = \infty$ is a regular point of R .

The set of all poles of the function R is said to be the *pole set* of the function R and is denoted by \mathcal{P}_R . The set of all zeros of the function R is said to be the *zero set* of the function R and is denoted by \mathcal{Z}_R .

Remark 11.2. Note that if R is a rational matrix function of simple structure then the inverse function R^{-1} is a rational matrix function of simple structure, as well, and $\mathcal{Z}_R = \mathcal{P}_{R^{-1}}$.

Below we formulate the "global" counterparts of Propositions 10.7 and 10.9 in order to characterize Fuchsian differential systems whose fundamental solutions are rational matrix functions of simple structure.

Theorem 11.3. *Let $R(z)$ be a rational matrix function of simple structure with the pole set \mathcal{P}_R and the zero set \mathcal{Z}_R . Then its logarithmic derivative² $Q_R(z)$ is a*

²See Remark 10.6.

rational function with the set of poles $\mathcal{P}_R \cup \mathcal{Z}_R$; all the poles of Q_R are simple. Furthermore, the function Q_R admits the additive decomposition

$$(11.1) \quad Q_R(z) = \sum_{t \in \mathcal{P}_R \cup \mathcal{Z}_R} \frac{Q_t}{z - t},$$

and its residues $Q_t \in \mathbb{C}^{m \times m}$ satisfy the following relations:

$$(11.2) \quad \sum_{t \in \mathcal{P}_R \cup \mathcal{Z}_R} Q_t = 0,$$

$$(11.3) \quad Q_t^2 = \begin{cases} -Q_t & \text{if } t \in \mathcal{P}_R, \\ Q_t & \text{if } t \in \mathcal{Z}_R, \end{cases}$$

$$(11.4) \quad Q_t C_t Q_t = \begin{cases} -C_t Q_t & \text{if } t \in \mathcal{P}_R, \\ Q_t C_t & \text{if } t \in \mathcal{Z}_R, \end{cases}$$

where

$$(11.5) \quad C_t = \sum_{\substack{t' \in \mathcal{P}_R \cup \mathcal{Z}_R \\ t' \neq t}} \frac{Q_{t'}}{t - t'}.$$

Proof. Since both functions R and R^{-1} are holomorphic in $\overline{\mathbb{C}} \setminus (\mathcal{P}_R \cup \mathcal{Z}_R)$, the logarithmic derivative Q_R is holomorphic there, as well. According to Proposition 10.7, each point of the set $\{\mathcal{P}_R \cup \mathcal{Z}_R\}$ is a simple pole of Q_R , hence we can write for Q_R the additive decomposition

$$Q_R(z) = Q_R(\infty) + \sum_{t \in \mathcal{P}_R \cup \mathcal{Z}_R} \frac{Q_t}{z - t},$$

where Q_t are the residues of Q_R . Since R is holomorphic at ∞ , the entries of its derivative R' decay as $o(|z|^{-1})$ when $z \rightarrow \infty$. The rate of decay for the logarithmic derivative Q_R is the same, because R^{-1} , too, is holomorphic at ∞ . Thus we obtain the additive decomposition (11.1) for Q_R and the relation (11.2) for the residues Q_t . Now the relations (11.3), (11.4) follow immediately from Proposition 10.7, once we observe that the matrix C_t given by (11.5) is but the constant term of the Laurent expansion of Q_R at its pole t . \square

Theorem 11.4. *Let \mathcal{P} and \mathcal{Z} be two finite disjoint subsets of the complex plane \mathbb{C} and let $Q(z)$ be a $\mathbb{C}^{m \times m}$ -valued rational function of the form*

$$(11.6) \quad Q(z) = \sum_{t \in \mathcal{P} \cup \mathcal{Z}} \frac{Q_t}{z - t},$$

where $Q_t \in \mathbb{C}^{m \times m}$. Let the matrices Q_t satisfy the relations

$$(11.7) \quad \sum_{t \in \mathcal{P} \cup \mathcal{Z}} Q_t = 0,$$

$$(11.8) \quad Q_t^2 = \begin{cases} -Q_t & \text{if } t \in \mathcal{P}, \\ Q_t & \text{if } t \in \mathcal{Z}, \end{cases}$$

$$(11.9) \quad Q_t C_t Q_t = \begin{cases} -C_t Q_t & \text{if } t \in \mathcal{P}, \\ Q_t C_t & \text{if } t \in \mathcal{Z}, \end{cases}$$

where

$$(11.10) \quad C_t = \sum_{\substack{t' \in \mathcal{P} \cup \mathcal{Z} \\ t' \neq t}} \frac{Q_{t'}}{t - t'}.$$

Let $R(z)$ be a fundamental solution of the Fuchsian differential system

$$(11.11) \quad R'(z) = Q(z)R(z).$$

Then R is a rational matrix function of simple structure such that

$$\mathcal{P}_R = \mathcal{P}, \quad \mathcal{Z}_R = \mathcal{Z}.$$

Proof. Since the condition (11.7) implies that the point ∞ is a regular point for the Fuchsian system (11.11), we may, without loss of generality, consider the fundamental solution R satisfying the initial condition $R(\infty) = I$. Then $R(z)$ is a matrix function, holomorphic (a priori, multi-valued) and invertible in the (multi-connected) set $\overline{\mathbb{C}} \setminus (\mathcal{P} \cup \mathcal{N})$. However, for $t \in \mathcal{P} \cup \mathcal{N}$ the function Q admits in a neighborhood of t the Laurent expansion

$$Q(z) = \frac{Q_t}{z - t} + C_t + o(1)$$

with the constant term C_t given by (11.10). The coefficients Q_t and C_t satisfy the relations (11.8), (11.9), hence by Proposition 10.9 the function R is meromorphic at t . Since this is true for every $t \in \mathcal{P} \cup \mathcal{N}$, the function R is rational (in particular, single-valued). Proposition 10.9 also implies that every $t \in \mathcal{P}$ (respectively, $t \in \mathcal{Z}$) is a simple pole (respectively, a zero) of the function R and a zero (respectively, a simple pole) of the inverse function R^{-1} . Therefore, R is a rational matrix function of simple structure with the pole set \mathcal{P} and the zero set \mathcal{Z} . \square

We close this section with the following useful

Lemma 11.5. *Let R be a rational matrix function of simple structure. For $t \in \mathcal{P}_R \cup \mathcal{Z}_R$ let R_t denote the residue of the function R at t if $t \in \mathcal{P}_R$, and the residue of the inverse function R^{-1} at t if $t \in \mathcal{Z}_R$. Then*

$$(11.12) \quad \sum_{t \in \mathcal{P}_R} \text{rank}(R_t) = \sum_{t \in \mathcal{Z}_R} \text{rank}(R_t).$$

Proof. Let us consider the logarithmic derivative Q_R of R . Its residues Q_t satisfy by Theorem 11.3 the relations (11.2) and (11.3). From (11.3) it follows that

$$\text{rank}(Q_t) = \begin{cases} -\text{trace}(Q_t) & \text{if } t \in \mathcal{P}_R, \\ \text{trace}(Q_t) & \text{if } t \in \mathcal{Z}_R. \end{cases}$$

But (11.2) implies

$$\sum_{t \in \mathcal{P}_R} \text{trace}(Q_t) + \sum_{t \in \mathcal{Z}_R} \text{trace}(Q_t) = 0,$$

hence

$$\sum_{t \in \mathcal{P}_R} \text{rank}(Q_t) = \sum_{t \in \mathcal{Z}_R} \text{rank}(Q_t).$$

Finally, by Proposition 10.7 (see (10.3), (10.5) there),

$$\text{rank}(R_t) = \text{rank}(Q_t), \quad \forall t \in \mathcal{P}_R \cup \mathcal{Z}_R.$$

Thus (11.12) holds. \square

12. GENERIC RATIONAL MATRIX FUNCTIONS

Definition 12.1. A $\mathbb{C}^{m \times m}$ -valued rational function $R(z)$ is said to be a *generic*³ rational matrix function if R is a rational matrix function of simple structure and all the residues of the functions R and R^{-1} have rank one.

Lemma 12.2. *Let R be a generic rational matrix function. Then the cardinalities of its pole and zero sets⁴ coincide:*

$$(12.1) \quad \#\mathcal{P}_R = \#\mathcal{Z}_R.$$

Proof. Since all the residues of R and R^{-1} are of rank one, the statement follows immediately from Lemma 11.5. \square

Let R be a $\mathbb{C}^{m \times m}$ -valued generic rational function. In what follows, we assume that R is normalized by

$$(12.2) \quad R(\infty) = I.$$

Let us order somehow the pole and zero sets of R :

$$(12.3) \quad \mathcal{P}_R = \{t_1, \dots, t_n\}, \quad \mathcal{Z}_R = \{t_{n+1}, \dots, t_{2n}\},$$

where $n = \#\mathcal{P}_R = \#\mathcal{Z}_R$. Then we can write for R and R^{-1} the additive decompositions

$$(12.4a) \quad R(z) = I + \sum_{k=1}^n \frac{R_k}{z - t_k},$$

$$(12.4b) \quad R^{-1}(z) = I + \sum_{k=n+1}^{2n} \frac{R_k}{z - t_k},$$

where for $1 \leq k \leq n$ (respectively, $n+1 \leq k \leq 2n$) we denote by R_k the residue of R (respectively, R^{-1}) at its pole t_k . Since each matrix R_k is of rank one, the representations (12.4) can be rewritten as

$$(12.5a) \quad R(z) = I + \sum_{k=1}^n f_k \frac{1}{z - t_k} g_k,$$

$$(12.5b) \quad R^{-1}(z) = I + \sum_{k=n+1}^{2n} f_k \frac{1}{z - t_k} g_k,$$

where for $1 \leq k \leq n$ (respectively, $n+1 \leq k \leq 2n$) $f_k \in \mathbb{C}^{m \times 1}$ and $g_k \in \mathbb{C}^{1 \times m}$ are the left and right semiresidues⁵ of R (respectively, R^{-1}) at t_k . Furthermore, we introduce two $n \times n$ diagonal matrices:

$$(12.6) \quad A_{\mathcal{P}} = \text{diag}(t_1, \dots, t_n), \quad A_{\mathcal{Z}} = \text{diag}(t_{n+1}, \dots, t_{2n}),$$

two $m \times n$ matrices :

$$(12.7) \quad F_{\mathcal{P}} = (f_1 \quad \dots \quad f_n), \quad F_{\mathcal{Z}} = (f_{n+1} \quad \dots \quad f_{2n}),$$

³In [Kats2] such functions are called "rational matrix functions in general position".

⁴See Definition 11.1.

⁵See Definition 10.1.

and two $n \times m$ matrices :

$$(12.8) \quad G_{\mathcal{P}} = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}, \quad G_{\mathcal{Z}} = \begin{pmatrix} g_{n+1} \\ \vdots \\ g_{2n} \end{pmatrix}.$$

The matrices $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ are said to be, respectively, the *pole* and *zero matrices* of R . The matrices $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ are said to be, respectively, the *left* and *right pole semiresidual matrices* of R . Analogously, the matrices $F_{\mathcal{Z}}$ and $G_{\mathcal{Z}}$ are said to be the *left* and *right zero semiresidual matrices* of R .

Remark 12.3. It should be mentioned that for a fixed ordering (12.3) of the pole and zero sets the pole and the zero matrices $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ are defined uniquely, and the semiresidual matrices $F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{Z}}, G_{\mathcal{Z}}$ are defined essentially uniquely, up to the transformation

$$(12.9a) \quad F_{\mathcal{P}} \rightarrow F_{\mathcal{P}} D_{\mathcal{P}}, \quad G_{\mathcal{P}} \rightarrow D_{\mathcal{P}}^{-1} G_{\mathcal{P}},$$

$$(12.9b) \quad F_{\mathcal{Z}} \rightarrow F_{\mathcal{Z}} D_{\mathcal{Z}}, \quad G_{\mathcal{Z}} \rightarrow D_{\mathcal{Z}}^{-1} G_{\mathcal{Z}},$$

where $D_{\mathcal{P}}, D_{\mathcal{Z}} \in \mathbb{C}^{n \times n}$ are arbitrary invertible diagonal matrices. Once the choice of the left pole semiresidual matrix $F_{\mathcal{P}}$ is fixed, the right pole semiresidual matrix $G_{\mathcal{P}}$ is determined uniquely, etc.

In terms of the matrices $A_{\mathcal{P}}, A_{\mathcal{Z}}, F_{\mathcal{P}}, G_{\mathcal{P}}, F_{\mathcal{Z}}, G_{\mathcal{Z}}$, the representations (12.5) take the following form:

$$(12.10a) \quad R(z) = I + F_{\mathcal{P}} (zI - A_{\mathcal{P}})^{-1} G_{\mathcal{P}},$$

$$(12.10b) \quad R^{-1}(z) = I + F_{\mathcal{Z}} (zI - A_{\mathcal{Z}})^{-1} G_{\mathcal{Z}}.$$

The representations (12.10) are not quite satisfactory for the following reasons. Firstly, in view of the identity $RR^{-1} = R^{-1}R = I$, the information contained in the pair of representations (12.10) is *redundant*: each of these representations determines the function R (and R^{-1}) uniquely. Secondly, if, for example, the diagonal matrix $A_{\mathcal{P}}$ and the matrices $F_{\mathcal{P}}, G_{\mathcal{P}}$ of appropriate dimensions are chosen arbitrarily then the rational function (12.10a) need not be generic. In our investigation we shall mainly use another version of the system representation of rational matrix functions, more suitable for application to linear differential equations. This is the so-called joint system representation (see [Kats2] for details and references) of the function $R(z)R^{-1}(w)$ of *two* independent variables z and w . A key role in the theory of the joint system representation is played by the *Lyapunov equations*. These are matricial equations of the form

$$(12.11) \quad UX - XV = Y,$$

where the matrices $U, V, Y \in \mathbb{C}^{n \times n}$ are given, and the matrix $X \in \mathbb{C}^{n \times n}$ is unknown. If the spectra of the matrices U and V are disjoint, then the Lyapunov equation (12.11) is uniquely solvable with respect to X for arbitrary right-hand side Y . The solution X can be expressed, for example, as the contour integral

$$(12.12) \quad X = \frac{1}{2\pi i} \oint_{\Gamma} (zI - U)^{-1} Y (zI - V)^{-1} dz,$$

where Γ is an arbitrary contour, such that the spectrum of U is inside Γ and the spectrum of V is outside Γ (see, for instance, Chapter I, Section 3 of the book [DaKr]).

With the generic rational function R we associate the following pair of Lyapunov equations (with unknown $S_{\mathcal{Z}\mathcal{P}}, S_{\mathcal{P}\mathcal{Z}} \in \mathbb{C}^{n \times n}$):

$$(12.13a) \quad A_{\mathcal{Z}}S_{\mathcal{Z}\mathcal{P}} - S_{\mathcal{Z}\mathcal{P}}A_{\mathcal{P}} = G_{\mathcal{Z}}F_{\mathcal{P}},$$

$$(12.13b) \quad A_{\mathcal{P}}S_{\mathcal{P}\mathcal{Z}} - S_{\mathcal{P}\mathcal{Z}}A_{\mathcal{Z}} = G_{\mathcal{P}}F_{\mathcal{Z}}.$$

Since the spectra of the pole and zero matrices $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ do not intersect (these are the pole and zero sets of R), the Lyapunov equations (12.13a) and (12.13b) are uniquely solvable. In fact, since the matrices $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ are diagonal, the solutions can be given explicitly (using the notation (12.6) – (12.8)):

$$(12.14) \quad S_{\mathcal{Z}\mathcal{P}} = \left(\frac{g_{n+i}f_j}{t_{n+i} - t_j} \right)_{i,j=1}^n, \quad S_{\mathcal{P}\mathcal{Z}} = \left(\frac{g_i f_{n+j}}{t_i - t_{n+j}} \right)_{i,j=1}^n.$$

The matrices $S_{\mathcal{Z}\mathcal{P}}$ and $S_{\mathcal{P}\mathcal{Z}}$ are said to be, respectively, the *zero-pole* and *pole-zero coupling matrices* of R .

Proposition 12.4. *Let $R(z)$ be a generic rational matrix function normalized by $R(\infty) = I$. Then*

- (1) *the coupling matrices $S_{\mathcal{Z}\mathcal{P}}$ and $S_{\mathcal{P}\mathcal{Z}}$ of R are mutually inverse:*

$$(12.15) \quad S_{\mathcal{Z}\mathcal{P}}S_{\mathcal{P}\mathcal{Z}} = S_{\mathcal{P}\mathcal{Z}}S_{\mathcal{Z}\mathcal{P}} = I;$$

- (2) *for the semiresidual matrices of R the following relations hold:*

$$(12.16) \quad G_{\mathcal{Z}} = -S_{\mathcal{Z}\mathcal{P}}G_{\mathcal{P}}, \quad F_{\mathcal{P}} = F_{\mathcal{Z}}S_{\mathcal{Z}\mathcal{P}};$$

- (3) *the function R admits the joint representation*

$$(12.17) \quad R(z)R^{-1}(\omega) = I + (z - \omega)F_{\mathcal{P}}(zI - A_{\mathcal{P}})^{-1}S_{\mathcal{Z}\mathcal{P}}^{-1}(\omega I - A_{\mathcal{Z}})^{-1}G_{\mathcal{Z}},$$

where $A_{\mathcal{P}}, A_{\mathcal{Z}}$ are the pole and zero matrices of R .

Proof. The proof of Proposition 12.4 can be found in [Kats2]. \square

Remark 12.5. Note that, since $R(\infty) = I$, one can recover from the joint representation (12.17) when $z \rightarrow \infty$ or $\omega \rightarrow \infty$ the separate representations

$$(12.18a) \quad R(z) = I - F_{\mathcal{P}}(zI - A_{\mathcal{P}})^{-1}S_{\mathcal{Z}\mathcal{P}}^{-1}G_{\mathcal{Z}},$$

$$(12.18b) \quad R^{-1}(\omega) = I + F_{\mathcal{P}}S_{\mathcal{Z}\mathcal{P}}^{-1}(\omega I - A_{\mathcal{Z}})^{-1}G_{\mathcal{Z}},$$

which, in view of (12.16), coincide with the representations (12.10).

Remark 12.6. In view of (12.15), (12.16), one can also write the joint representation for R in terms of the matrices $F_{\mathcal{Z}}, G_{\mathcal{P}}$ and the solution $S_{\mathcal{P}\mathcal{Z}}$ of the Lyapunov equation (12.13b):

$$(12.19) \quad R(z)R^{-1}(\omega) = I - (z - \omega)F_{\mathcal{Z}}S_{\mathcal{P}\mathcal{Z}}^{-1}(zI - A_{\mathcal{P}})^{-1}S_{\mathcal{P}\mathcal{Z}}(\omega I - A_{\mathcal{Z}})^{-1}S_{\mathcal{P}\mathcal{Z}}^{-1}G_{\mathcal{P}}.$$

Thus we may conclude that the pole and zero sets together with a pair of the semiresidual matrices (either right pole and left zero or left pole and right zero) determine the normalized generic rational function R *uniquely*.

Remark 12.7. The theory of system representations for rational matrix functions with prescribed zero and pole structures first appeared in [GKLR], and was further developed in [BGR1], [BGR2], and [BGRa].

The joint representations (12.17), (12.19) suggest that this theory can be applied to the investigation of families of rational functions parameterized by the zeros' and poles' loci and the corresponding⁶ deformations of linear differential systems. The version of this theory adapted for such applications was presented in [Kats1] and [Kats2]. Also in [Kats2] one can find some historical remarks and a list of references.

Proposition 12.8. *Let $R(z)$ be a generic rational matrix function normalized by $R(\infty) = I$. Then its logarithmic derivative⁷ admits the representation:*

$$(12.20) \quad R'(z)R^{-1}(z) = F_{\mathcal{P}}(zI - A_{\mathcal{P}})^{-1}S_{\mathcal{Z}\mathcal{P}}^{-1}(zI - A_{\mathcal{Z}})^{-1}G_{\mathcal{Z}}$$

where $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ are the pole and zero matrices of R ; $F_{\mathcal{P}}$ and $G_{\mathcal{Z}}$ are the left pole and right zero semiresidual matrices of R ; $S_{\mathcal{Z}\mathcal{P}}$ is the zero-pole coupling matrix of R .

Proof. Differentiating (12.17) with respect to z , we obtain

$$R'(z)R^{-1}(\omega) = F_{\mathcal{P}}(I - (z - \omega)(zI - A_{\mathcal{P}})^{-1})(zI - A_{\mathcal{P}})^{-1}S_{\mathcal{Z}\mathcal{P}}^{-1}(\omega I - A_{\mathcal{Z}})^{-1}G_{\mathcal{Z}}.$$

Now set $\omega = z$ to obtain (12.20). \square

Remark 12.9. The representation (12.20) for the logarithmic derivative Q_R of the normalized generic rational matrix function R can also be rewritten in terms of the matrices $F_{\mathcal{Z}}, G_{\mathcal{P}}$ and the solution $S_{\mathcal{P}\mathcal{Z}}$ of the Lyapunov equation (12.13b) (see Remark 12.6):

$$(12.21) \quad R'(z)R^{-1}(z) = -F_{\mathcal{Z}}S_{\mathcal{P}\mathcal{Z}}^{-1}(zI - A_{\mathcal{P}})^{-1}S_{\mathcal{P}\mathcal{Z}}(zI - A_{\mathcal{Z}})^{-1}S_{\mathcal{P}\mathcal{Z}}^{-1}G_{\mathcal{P}}.$$

13. GENERIC RATIONAL MATRIX FUNCTIONS WITH PRESCRIBED LOCAL DATA

In the previous section we discussed the question, how to represent a generic rational function R in terms of its local data (the pole and zero sets and the residues). The main goal of this section is to construct a (normalized) generic rational matrix $R(z)$ function with *prescribed* local data. In view of Proposition 12.4, Remark 12.6 and Remark 12.3, such data should be given in the form of two diagonal matrices of the same dimension (the pole and zero matrices) and two semiresidual matrices of appropriate dimensions (either right pole and left zero or right zero and left pole)⁸.

Thus we consider the following

Problem 13.1. Let two diagonal matrices

$$A_{\mathcal{P}} = \text{diag}(t_1, \dots, t_n), \quad A_{\mathcal{Z}} = \text{diag}(t_{n+1}, \dots, t_{2n}), \quad t_i \neq t_j \text{ unless } i = j,$$

and two matrices: $F \in \mathbb{C}^{m \times n}$, $G \in \mathbb{C}^{n \times m}$, be given.

\mathcal{ZP} -version: Find a generic $\mathbb{C}^{m \times m}$ -valued rational function $R(z)$ such that

- (1) $R(\infty) = I$;
- (2) the matrices $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ are, respectively, the pole and zero matrices of R ;
- (3) the matrix F is the left pole semiresidual matrix $F_{\mathcal{P}}$ of R : $F = F_{\mathcal{P}}$;
- (4) the matrix G is the right zero semiresidual matrix $G_{\mathcal{Z}}$ of R : $G = G_{\mathcal{Z}}$.

⁶See Propositions 11.3, 11.4.

⁷See Remark 10.6.

⁸Here we use the terminology introduced in Section 12.

\mathcal{PZ} -version: Find a generic $\mathbb{C}^{m \times m}$ -valued rational function $R(z)$ such that

- (1) $R(\infty) = I$;
- (2) the matrices $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$ are, respectively, the pole and zero matrices of R ;
- (3) the matrix F is the left zero semiresidual matrix $F_{\mathcal{Z}}$ of R : $F = F_{\mathcal{Z}}$;
- (4) the matrix G is the right pole semiresidual matrix $G_{\mathcal{P}}$ of R : $G = G_{\mathcal{P}}$.

Proposition 13.2.

- (1) *The \mathcal{ZP} -version of Problem 13.1 is solvable if and only if the solution S of the Lyapunov equation*

$$(13.1) \quad A_{\mathcal{Z}}S - SA_{\mathcal{P}} = GF$$

is an invertible matrix.

- (2) *The \mathcal{PZ} -version of Problem 13.1 is solvable if and only if the solution S of the Lyapunov equation*

$$(13.2) \quad A_{\mathcal{P}}S - SA_{\mathcal{Z}} = GF$$

is an invertible matrix.

Proof. The proof of Proposition 13.2 can be found in [Kats2]. Here we would like to note that the solutions of the Lyapunov equations (13.1) and (13.2) can be written explicitly as, respectively,

$$(13.3) \quad S = \left(\frac{g_i f_j}{t_{n+i} - t_j} \right)_{i,j=1}^n \quad \text{and} \quad S = \left(\frac{g_i f_j}{t_i - t_{n+j}} \right)_{i,j=1}^n,$$

where g_i is the i -th row of the matrix G and f_j is the j -th column of the matrix F . Note also that the necessity of S being invertible in both cases follows from Proposition 12.4. \square

In view of Proposition 13.2, we propose the following terminology:

Definition 13.3. Let $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ be the given data of Problem 13.1. Then:

- (1) the solution S of the Lyapunov equation

$$(13.4) \quad A_{\mathcal{Z}}S - SA_{\mathcal{P}} = GF$$

is said to be the \mathcal{ZP} -coupling matrix related to the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$;

- (2) the solution S of the Lyapunov equation

$$(13.5) \quad A_{\mathcal{P}}S - SA_{\mathcal{Z}} = GF$$

is said to be the \mathcal{PZ} -coupling matrix related to the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$;

- (3) the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ are said to be \mathcal{ZP} -admissible if the \mathcal{ZP} -coupling matrix related to this data is invertible;
- (4) the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ are said to be \mathcal{PZ} -admissible if the \mathcal{PZ} -coupling matrix related to this data is invertible.

Proposition 13.4. Let $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ be the given data of Problem 13.1.

- (1) *If the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ are \mathcal{ZP} -admissible then the \mathcal{ZP} -version of Problem 13.1 has the unique solution $R(z)$ given by*

$$(13.6) \quad R(z) = I - F(I - A_{\mathcal{P}})^{-1}S^{-1}G,$$

where S is the \mathcal{ZP} -coupling matrix related to the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$. The logarithmic derivative of R is given by

$$(13.7) \quad R'(z)R^{-1}(z) = F(zI - A_{\mathcal{P}})^{-1}S^{-1}(zI - A_{\mathcal{Z}})^{-1}G.$$

- (2) If the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ are \mathcal{PZ} -admissible then the \mathcal{PZ} -version of Problem 13.1 has the unique solution $R(z)$ given by

$$(13.8) \quad R(z) = I + FS^{-1}(zI - A_{\mathcal{P}})^{-1}G,$$

where S is the \mathcal{PZ} -coupling matrix related to the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$. The logarithmic derivative of R is given by

$$(13.9) \quad R'(z)R^{-1}(z) = -FS^{-1}(zI - A_{\mathcal{P}})^{-1}S(zI - A_{\mathcal{Z}})^{-1}S^{-1}G.$$

Proof. If the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ are \mathcal{ZP} -admissible then, according to Proposition 13.2, the \mathcal{ZP} -version of Problem 13.1 has a solution R . Then the \mathcal{ZP} -coupling matrix S related to the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ is the zero-pole coupling matrix of R . According to Proposition 12.4, the function R admits the representation (13.6) and hence⁹ is determined uniquely. Now the representation (13.7) for the logarithmic derivative follows from Proposition 12.8.

Analogous considerations hold also in the case when the data $A_{\mathcal{P}}, A_{\mathcal{Z}}, F, G$ are \mathcal{PZ} -admissible (see Remarks 12.6, 12.9). \square

It was already mentioned (see Remark 12.3) that in the definition of the semiresidual matrices there is a certain freedom. Accordingly, certain equivalency classes rather than individual matrices F, G should serve as data for Problem 13.1. The appropriate definitions are similar to the definition of the complex projective space \mathbb{P}^{k-1} as the space of equivalency classes of the set $\mathbb{C}^k \setminus \{0\}$ (two vectors $h', h'' \in \mathbb{C}^k \setminus \{0\}$ are declared to be *equivalent* if h', h'' are *proportional*, i.e. $h'' = dh'$ for some $d \in \mathbb{C}_*$).

Definition 13.5. (1) Let $\mathbb{C}_{*,c}^{m \times n}$ denote the set of $m \times n$ matrices which have no zero columns. Two matrices $F', F'' \in \mathbb{C}_{*,c}^{m \times n}$ are declared to be equivalent: $F' \stackrel{c}{\sim} F''$, if

$$(13.10) \quad F'' = F'D_c,$$

where D_c is a diagonal invertible matrix.

The space $\mathbb{P}_c^{(m-1) \times n}$ is a factor-set of the set $\mathbb{C}_{*,c}^{m \times n}$ modulo the equivalency relation $\stackrel{c}{\sim}$.

- (2) Let $\mathbb{C}_{*,r}^{n \times m}$ denote the set of $n \times m$ matrices which have no zero rows. Two matrices $G', G'' \in \mathbb{C}_{*,r}^{n \times m}$ are declared to be equivalent: $G' \stackrel{r}{\sim} G''$, if

$$(13.11) \quad G'' = D_r G',$$

where D_r is a diagonal invertible matrix.

The space $\mathbb{P}_r^{n \times (m-1)}$ is a factor-set of the set $\mathbb{C}_{*,r}^{n \times m}$ modulo the equivalency relation $\stackrel{r}{\sim}$.

The factor spaces $\mathbb{P}_c^{(m-1) \times n}$ and $\mathbb{P}_r^{n \times (m-1)}$ inherit topology from the spaces $\mathbb{C}_{*,c}^{m \times n}$ and $\mathbb{C}_{*,r}^{n \times m}$, respectively. They can be provided naturally with the structure of complex manifolds.

If F' and F'' are two $\stackrel{c}{\sim}$ -equivalent $m \times n$ matrices, and G' and G'' are two $\stackrel{r}{\sim}$ -equivalent $n \times m$ matrices, then the solutions S', S'' of the Lyapunov equation

⁹See Remark 12.5.

(13.1) with F', G', F'', G'' , substituted instead of F, G , and the same $A_{\mathcal{P}}, A_{\mathcal{Z}}$ are related by

$$(13.12) \quad S'' = D_r S' D_c,$$

where D_c, D_r are the invertible diagonal matrices, which appear in (13.10), (13.11). Similar result holds also for the Lyapunov equations (13.2).

However, since diagonal matrices commute, the expressions on the right-hand side of (13.6) will not be changed if we replace the matrices F, G, S with the matrices $FD_c, D_r G, D_r S D_c$, respectively.

Thus, the following result holds:

Proposition 13.6. *Given $A_{\mathcal{P}}$ and $A_{\mathcal{Z}}$, solution of Problem 13.1 depends not on the matrices F, G themselves but on their equivalency classes in $\mathbb{P}_c^{(m-1) \times n}, \mathbb{P}_r^{n \times (m-1)}$.*

Remark 13.7. In view of Remark 12.3, if R is a generic rational matrix function then its left and right pole semiresidual matrix $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ can be considered *separately* as elements of the sets $\mathbb{P}_c^{(m-1) \times n}$ and $\mathbb{P}_r^{n \times (m-1)}$, respectively. However, *simultaneously* the matrices $F_{\mathcal{P}}$ and $G_{\mathcal{P}}$ can not be considered so. The same holds for the pair of the zero semiresidual matrices, as well.

14. HOLOMORPHIC FAMILIES OF GENERIC RATIONAL MATRIX FUNCTIONS

Definition 14.1. Let \mathcal{D} be a domain¹⁰ in the space \mathbb{C}_*^{2n} and for every $\mathbf{t} = (t_1, \dots, t_{2n}) \in \mathcal{D}$ let $R(z, \mathbf{t})$ be a generic $\mathbb{C}^{m \times m}$ -valued rational function of z with the pole and zero matrices

$$(14.1) \quad A_{\mathcal{P}}(\mathbf{t}) = \text{diag}(t_1, \dots, t_n), \quad A_{\mathcal{Z}}(\mathbf{t}) = \text{diag}(t_{n+1}, \dots, t_{2n}).$$

Assume that for every $\mathbf{t}^0 \in \mathcal{D}$ and for every fixed $z \in \mathbb{C} \setminus \{t_1^0, \dots, t_{2n}^0\}$ the function $R(z, \mathbf{t})$ is holomorphic with respect to \mathbf{t} in a neighborhood of \mathbf{t}^0 . Assume also that

$$(14.2) \quad R(\infty, \mathbf{t}) \equiv I.$$

Then the family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ is said to be a *normalized holomorphic family* of generic rational functions parameterized by the pole and zero loci.

Given a normalized holomorphic family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ of generic rational functions parameterized by the pole and zero loci, we can write for each fixed $\mathbf{t} \in \mathcal{D}$ the following representations for the functions $R(z, \mathbf{t})$, $R^{-1}(z, \mathbf{t})$ and the logarithmic derivative

$$(14.3) \quad Q_R(z, \mathbf{t}) \stackrel{\text{def}}{=} \frac{\partial R(z, \mathbf{t})}{\partial z} R^{-1}(z, \mathbf{t})$$

(see (11.1), (12.4)):

$$(14.4a) \quad R(z, \mathbf{t}) = I + \sum_{k=1}^n \frac{R_k(\mathbf{t})}{z - t_k},$$

$$(14.4b) \quad R^{-1}(z, \mathbf{t}) = I + \sum_{k=n+1}^{2n} \frac{R_k(\mathbf{t})}{z - t_k},$$

$$(14.4c) \quad Q_R(z, \mathbf{t}) = \sum_{k=1}^{2n} \frac{Q_k(\mathbf{t})}{z - t_k}.$$

¹⁰One can also consider a Riemann domain over \mathbb{C}_*^{2n} (see Definition 5.4.4 in [Hör]).

The residues $R_k(\mathbf{t})$, $Q_k(\mathbf{t})$, considered as functions of \mathbf{t} , are defined in the whole domain \mathcal{D} . It is not hard to see that these functions are holomorphic in \mathcal{D} :

Lemma 14.2. *Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of generic rational functions, parameterized by the pole and zero loci. For each fixed $\mathbf{t} \in \mathcal{D}$ and $1 \leq k \leq n$ (respectively, $n+1 \leq k \leq 2n$) let $R_k(\mathbf{t})$ be the residue of the rational function $R(\cdot, \mathbf{t})$ (respectively, $R^{-1}(\cdot, \mathbf{t})$) at its pole t_k . Likewise, for each fixed $\mathbf{t} \in \mathcal{D}$ and $1 \leq k \leq 2n$ let $Q_k(\mathbf{t})$ be the residue of the logarithmic derivative $Q_R(\cdot, \mathbf{t})$ at its pole t_k . Then $R_k(\mathbf{t})$, $Q_k(\mathbf{t})$ considered as functions of \mathbf{t} are holomorphic in \mathcal{D} .*

Proof. Let us choose an arbitrary $\mathbf{t}^0 \in \mathcal{D}$ and n pairwise distinct points z_1, \dots, z_n in $\mathbb{C} \setminus \{t_1^0, \dots, t_n^0\}$. From the expansion (14.4a) we derive the following system of linear equations with respect to the residue matrices $R_k(\mathbf{t})$:

$$(14.5) \quad \sum_{k=1}^n \frac{R_k(\mathbf{t})}{z_\ell - t_k} = R(z_\ell, \mathbf{t}) - I, \quad \ell = 1, \dots, n.$$

The matrices $R(z_\ell, \mathbf{t}) - I$ on the right-hand side of the system (14.5) are holomorphic with respect to \mathbf{t} in a neighborhood of \mathbf{t}^0 . The determinant of this linear system

$$\Delta(\mathbf{t}) = \det \left(\frac{1}{z_\ell - t_k} \right)_{1 \leq \ell, k \leq n}$$

is holomorphic in a neighborhood of \mathbf{t}^0 , as well. In fact, the determinant $\Delta(\mathbf{t})$ (known as the *Cauchy determinant*) can be calculated explicitly (see, for example, [[PS], part VII, Section 1, No. 3]):

$$\Delta(\mathbf{t}) = \pm \frac{\prod_{1 \leq p < q \leq n} (z_p - z_q)(t_p - t_q)}{\prod_{\ell, k=1}^n (z_\ell - t_k)}.$$

In particular, $\Delta(\mathbf{t}^0) \neq 0$. Hence, for $k = 1, \dots, n$, the functions $R_k(\mathbf{t})$ are holomorphic in a neighborhood of \mathbf{t}^0 . Since this is true for any $\mathbf{t}^0 \in \mathcal{D}$, these functions are holomorphic in the whole domain \mathcal{D} . The proof for $R_k(\mathbf{t})$, when $n+1 \leq k \leq 2n$, and for $Q_k(\mathbf{t})$ is completely analogous. \square

Remark 14.3. Note that, on the one hand, the functions $R(z, \mathbf{t})$, $R^{-1}(z, \mathbf{t})$ are rational with respect to z , and hence are holomorphic with respect to z in $\mathbb{C} \setminus \{t_1, \dots, t_{2n}\}$. On the other hand, for every fixed $z \in \mathbb{C}$, these functions are holomorphic with respect to \mathbf{t} in $\mathcal{D} \setminus \bigcup_{k=1}^{2n} \{\mathbf{t} : t_k = z\}$. Thus, by Hartogs theorem (see for example [Shab], Chapter I, sections 2.3, 2.6), the matrix functions $R(z, \mathbf{t})$, $R^{-1}(z, \mathbf{t})$ are *jointly* holomorphic in the variables z, \mathbf{t} outside the singular set $\bigcup_{k=1}^{2n} \{x, \mathbf{t} : t_k = x\}$. In view of Lemma 14.2, the same conclusion follows from the representations (14.4a), (14.4b).

In order to employ the joint system representation techniques described in Sections 12 and 13 in the present setting, we have to establish the holomorphy not only of the residues but also of the semiresidues of $R(\cdot, \mathbf{t})$. In view of Remarks 10.2 and 12.3, we have a certain freedom in definition of the semiresidues and the semiresidual matrices. Thus one should take care in choosing the semiresidual matrices of $R(\cdot, \mathbf{t})$ for each fixed \mathbf{t} in order to obtain holomorphic functions of \mathbf{t} . In general, it is possible to define the holomorphic semiresidues only locally (we refer the reader

to Appendix B of the present paper where the global holomorphic factorization of a matrix function of rank one is discussed).

Lemma 14.4. *Let $M(\mathbf{t})$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic in a domain \mathcal{D} of \mathbb{C}^N , and let*

$$(14.6) \quad \text{rank } M(\mathbf{t}) = 1 \quad \forall \mathbf{t} \in \mathcal{D}.$$

Then there exist a finite open covering $\{\mathcal{U}_p\}_{p=1}^m$ of \mathcal{D} , a collection $\{f_p(\mathbf{t})\}_{p=1}^m$ of $\mathbb{C}^{m \times 1}$ -valued functions and a collection $\{g_p(\mathbf{t})\}_{p=1}^m$ of $\mathbb{C}^{1 \times m}$ -valued functions satisfying the following conditions.

- (1) *For $p = 1, \dots, m$ the functions $f_p(\mathbf{t})$ and $g_p(\mathbf{t})$ are holomorphic in \mathcal{U}_p .*
- (2) *Whenever $\mathcal{U}_{p'} \cap \mathcal{U}_{p''} \neq \emptyset$, there exists a (scalar) function $\varphi_{p',p''}(\mathbf{t})$, holomorphic and invertible in $\mathcal{U}_{p'} \cap \mathcal{U}_{p''}$, such that for every $\mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''}$*

$$(14.7) \quad f_{p''}(\mathbf{t}) = f_{p'}(\mathbf{t})\varphi_{p',p''}(\mathbf{t}), \quad g_{p''}(\mathbf{t}) = \varphi_{p',p''}^{-1}(\mathbf{t})g_{p'}(\mathbf{t}).$$

- (3) *For $p = 1, \dots, m$ the function $M(\mathbf{t})$ admits the factorization*

$$(14.8) \quad M(\mathbf{t}) = f_p(\mathbf{t})g_p(\mathbf{t}), \quad \mathbf{t} \in \mathcal{U}_p.$$

Proof. Let $f_p(\mathbf{t})$ be the p -th column of the matrix $M(\mathbf{t})$ and let

$$(14.9) \quad \mathcal{U}_p = \{\mathbf{t} \in \mathcal{D} : f_p(\mathbf{t}) \neq 0\}, \quad p = 1, \dots, m.$$

Then, in view of (14.6) and (14.9), $\{\mathcal{U}_p\}_{p=1}^m$ is an open covering of \mathcal{D} . Furthermore, from (14.6) and (14.9) it follows that for $1 \leq p, q \leq m$ there exists a unique (scalar) function $\varphi_{p,q}(\mathbf{t})$, holomorphic in \mathcal{U}_p , such that

$$(14.10) \quad f_q(\mathbf{t}) = f_p(\mathbf{t})\varphi_{p,q}(\mathbf{t}), \quad \mathbf{t} \in \mathcal{U}_p.$$

Now define $g_p(\mathbf{t})$ as

$$(14.11) \quad g_p(\mathbf{t}) = (\varphi_{p,1}(\mathbf{t}) \quad \dots \quad \varphi_{p,m}(\mathbf{t})), \quad \mathbf{t} \in \mathcal{U}_p.$$

Then the function $g_p(\mathbf{t})$ is holomorphic in \mathcal{U}_p and, according to (14.10), the factorization (14.8) holds for these $f_p(\mathbf{t})$ and $g_p(\mathbf{t})$. (14.10) also implies that whenever $\mathcal{U}_{p'} \cap \mathcal{U}_{p''} \neq \emptyset$ we have

$$\varphi_{p'',p'}(\mathbf{t})\varphi_{p',k}(\mathbf{t}) = \varphi_{p'',k}(\mathbf{t}), \quad \mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''}, \quad 1 \leq k \leq m.$$

In particular,

$$\varphi_{p'',p'}(\mathbf{t})\varphi_{p',p''}(\mathbf{t}) = 1, \quad \mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''},$$

and (14.7) follows. \square

Theorem 14.5. *Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions parameterized by the pole and zero loci. Then there exist a finite open covering¹¹ $\{\mathcal{D}_\alpha\}_{\alpha \in \mathfrak{A}}$ of \mathcal{D} , two collections $\{F_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{F_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ of $\mathbb{C}^{m \times n}$ -valued functions and two collections $\{G_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ of $\mathbb{C}^{n \times m}$ -valued functions satisfying the following conditions.*

- (1) *For each $\alpha \in \mathfrak{A}$ the functions $F_{\mathcal{P},\alpha}(\mathbf{t})$, $F_{\mathcal{Z},\alpha}(\mathbf{t})$, $G_{\mathcal{P},\alpha}(\mathbf{t})$, $G_{\mathcal{Z},\alpha}(\mathbf{t})$ are holomorphic in \mathcal{D}_α .*

¹¹The index α runs over a finite indexing set \mathfrak{A} .

- (2) Whenever $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''} \neq \emptyset$, there exist diagonal matrix functions $D_{\mathcal{P},\alpha',\alpha''}(\mathbf{t})$, $D_{\mathcal{Z},\alpha',\alpha''}(\mathbf{t})$, holomorphic and invertible in $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$, such that for every $\mathbf{t} \in \mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$

$$(14.12a) \quad F_{\mathcal{P},\alpha''}(\mathbf{t}) = F_{\mathcal{P},\alpha'}(\mathbf{t})D_{\mathcal{P},\alpha',\alpha''}(\mathbf{t}), \quad G_{\mathcal{P},\alpha''}(\mathbf{t}) = D_{\mathcal{P},\alpha',\alpha''}^{-1}(\mathbf{t})G_{\mathcal{P},\alpha'}(\mathbf{t}),$$

$$(14.12b) \quad F_{\mathcal{Z},\alpha''}(\mathbf{t}) = F_{\mathcal{Z},\alpha'}(\mathbf{t})D_{\mathcal{Z},\alpha',\alpha''}(\mathbf{t}), \quad G_{\mathcal{Z},\alpha''}(\mathbf{t}) = D_{\mathcal{Z},\alpha',\alpha''}^{-1}(\mathbf{t})G_{\mathcal{Z},\alpha'}(\mathbf{t}).$$

- (3) For each $\alpha \in \mathfrak{A}$ and $\mathbf{t} \in \mathcal{D}_\alpha$ the matrices $F_{\mathcal{P},\alpha}(\mathbf{t})$, $F_{\mathcal{Z},\alpha}(\mathbf{t})$, $G_{\mathcal{P},\alpha}(\mathbf{t})$, $G_{\mathcal{Z},\alpha}(\mathbf{t})$ are, respectively, the left pole, left zero, right pole, right zero semiresidual matrices of the generic rational function $R(\cdot, \mathbf{t})$, i.e. the representations

$$(14.13a) \quad R(z, \mathbf{t}) = I + F_{\mathcal{P},\alpha}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}G_{\mathcal{P},\alpha}(\mathbf{t}),$$

$$(14.13b) \quad R^{-1}(z, \mathbf{t}) = I + F_{\mathcal{Z},\alpha}(\mathbf{t})(zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1}G_{\mathcal{Z},\alpha}(\mathbf{t})$$

hold true for all $\mathbf{t} \in \mathcal{D}_\alpha$.

Proof. Let $R_k(\mathbf{t})$, $k = 1, \dots, 2n$, be the holomorphic residue functions as in Lemma 14.2. Since for every fixed \mathbf{t} the rational function $R(\cdot, \mathbf{t})$ is generic, each matrix $R_k(\mathbf{t})$ is of rank one. Hence there exists a finite open covering $\{\mathcal{U}_{k,p}\}_{p=1}^m$ of \mathcal{D} , such that in each open set $\mathcal{U}_{k,p}$ the function $R_k(\mathbf{t})$ admits the factorization $R_k(\mathbf{t}) = f_{k,p}(\mathbf{t})g_{k,p}(\mathbf{t})$ as in Lemma 14.4. Now it suffices to define \mathfrak{A} as the set of $2n$ -tuples (p_1, \dots, p_{2n}) such that $\cap_{k=1}^{2n} \mathcal{U}_{k,p_k} \neq \emptyset$, the open covering $\{\mathcal{D}_\alpha\}_{\alpha \in \mathfrak{A}}$ of \mathcal{D} by

$$\mathcal{D}_{(p_1, \dots, p_{2n})} = \bigcap_{k=1}^{2n} \mathcal{U}_{k,p_k},$$

and the collections $\{F_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{F_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ by

$$F_{\mathcal{P},(p_1, \dots, p_{2n})}(\mathbf{t}) = (f_{1,p_1}(\mathbf{t}) \quad \dots \quad f_{n,p_n}(\mathbf{t})),$$

$$F_{\mathcal{Z},(p_1, \dots, p_{2n})}(\mathbf{t}) = (f_{n+1,p_{n+1}}(\mathbf{t}) \quad \dots \quad f_{2n,p_{2n}}(\mathbf{t})),$$

$$G_{\mathcal{P},(p_1, \dots, p_{2n})}(\mathbf{t}) = \begin{pmatrix} g_{1,p_1}(\mathbf{t}) \\ \vdots \\ g_{n,p_n}(\mathbf{t}) \end{pmatrix}, \quad G_{\mathcal{Z},(p_1, \dots, p_{2n})}(\mathbf{t}) = \begin{pmatrix} g_{n+1,p_{n+1}}(\mathbf{t}) \\ \vdots \\ g_{2n,p_{2n}}(\mathbf{t}) \end{pmatrix}.$$

□

Definition 14.6. Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions parameterized by the pole and zero loci. Let a finite open covering $\{\mathcal{D}_\alpha\}_{\alpha \in \mathfrak{A}}$ of \mathcal{D} , collections $\{F_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{F_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ of $\mathbb{C}^{m \times n}$ -valued functions and collections $\{G_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ of $\mathbb{C}^{n \times m}$ -valued functions satisfy the conditions 1. – 3. of Theorem 14.5. Then the collections $\{F_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{F_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ are said to be the collections of, respectively, the left pole, left zero, right pole, right zero semiresidual functions related to the family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$.

Now we can tackle the problem of recovery of a holomorphic family of generic matrix functions from the semiresidual data. Once again, let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions, parameterized by the pole and zero loci. Let collections $\{F_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{F_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{P},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_{\mathcal{Z},\alpha}(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ be the collections of the semiresidual matrices related to the family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$. Then for each $\alpha \in \mathfrak{A}$

and a fixed $\mathbf{t} \in \mathcal{D}_\alpha$ the matrices $A_{\mathcal{P}}(\mathbf{t})$, $A_{\mathcal{Z}}(\mathbf{t})$ are the pole and zero matrices of the generic rational function $R(z, \mathbf{t})$, and $F_{\mathcal{P}, \alpha}(\mathbf{t})$, $F_{\mathcal{Z}, \alpha}(\mathbf{t})$, $G_{\mathcal{P}, \alpha}(\mathbf{t})$, $G_{\mathcal{Z}, \alpha}(\mathbf{t})$ are, respectively, the left pole, left zero, right pole, right zero semiresidual matrices of $R(\cdot, \mathbf{t})$. The appropriate coupling matrices $S_{\mathcal{ZP}, \alpha}(\mathbf{t})$, $S_{\mathcal{PZ}, \alpha}(\mathbf{t})$ satisfying the Lyapunov equations

$$(14.14a) \quad A_{\mathcal{Z}}(\mathbf{t})S_{\mathcal{ZP}, \alpha}(\mathbf{t}) - S_{\mathcal{ZP}, \alpha}(\mathbf{t})A_{\mathcal{P}}(\mathbf{t}) = G_{\mathcal{Z}, \alpha}(\mathbf{t})F_{\mathcal{P}, \alpha}(\mathbf{t}),$$

$$(14.14b) \quad A_{\mathcal{P}}(\mathbf{t})S_{\mathcal{PZ}, \alpha}(\mathbf{t}) - S_{\mathcal{PZ}, \alpha}(\mathbf{t})A_{\mathcal{Z}}(\mathbf{t}) = G_{\mathcal{P}, \alpha}(\mathbf{t})F_{\mathcal{Z}, \alpha}(\mathbf{t})$$

are given by

$$(14.15) \quad S_{\mathcal{ZP}, \alpha}(\mathbf{t}) = \left(\frac{g_{n+i, \alpha}(\mathbf{t})f_{j, \alpha}(\mathbf{t})}{t_{n+i} - t_j} \right)_{i, j=1}^n, \quad S_{\mathcal{PZ}, \alpha}(\mathbf{t}) = \left(\frac{g_{i, \alpha}(\mathbf{t})f_{n+j, \alpha}(\mathbf{t})}{t_i - t_{n+j}} \right)_{i, j=1}^n,$$

where for $1 \leq k \leq n$ $g_{k, \alpha}(\mathbf{t})$ is the k -th row of $G_{\mathcal{P}, \alpha}(\mathbf{t})$, $g_{n+k, \alpha}(\mathbf{t})$ is the k -th row of $G_{\mathcal{Z}, \alpha}(\mathbf{t})$, $f_{k, \alpha}(\mathbf{t})$ is the k -th column of $F_{\mathcal{P}, \alpha}(\mathbf{t})$, $f_{n+k, \alpha}(\mathbf{t})$ is the k -th column of $F_{\mathcal{Z}, \alpha}(\mathbf{t})$ (compare with similar expressions (12.7), (12.8), (12.14)). From the explicit expressions (14.15) it is evident that $S_{\mathcal{ZP}, \alpha}(\mathbf{t})$, $S_{\mathcal{PZ}, \alpha}(\mathbf{t})$, considered as functions of \mathbf{t} , are holomorphic in \mathcal{D}_α . According to Proposition 12.4, the functions $S_{\mathcal{ZP}, \alpha}(\mathbf{t})$, $S_{\mathcal{PZ}, \alpha}(\mathbf{t})$ are mutually inverse:

$$(14.16) \quad S_{\mathcal{ZP}, \alpha}(\mathbf{t})S_{\mathcal{PZ}, \alpha}(\mathbf{t}) = S_{\mathcal{PZ}, \alpha}(\mathbf{t})S_{\mathcal{ZP}, \alpha}(\mathbf{t}) = I, \quad \mathbf{t} \in \mathcal{D}_\alpha,$$

and the following relations hold:

$$(14.17) \quad G_{\mathcal{Z}, \alpha}(\mathbf{t}) = -S_{\mathcal{ZP}, \alpha}(\mathbf{t})G_{\mathcal{P}, \alpha}(\mathbf{t}), \quad F_{\mathcal{P}, \alpha}(\mathbf{t}) = F_{\mathcal{Z}, \alpha}(\mathbf{t})S_{\mathcal{ZP}, \alpha}(\mathbf{t}).$$

Furthermore, for $\mathbf{t} \in \mathcal{D}_\alpha$ the function $R(z, \mathbf{t})$ admits the representation

$$(14.18) \quad R(z, \mathbf{t})R^{-1}(\omega, \mathbf{t}) = I + (z - \omega)F_{\mathcal{P}, \alpha}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_{\mathcal{ZP}, \alpha}^{-1}(\mathbf{t})(\omega I - A_{\mathcal{Z}}(\mathbf{t}))^{-1}G_{\mathcal{Z}, \alpha}(\mathbf{t}),$$

and, in view of Proposition 12.8, its logarithmic derivative with respect to z admits the representation

$$(14.19) \quad \frac{\partial R(z, \mathbf{t})}{\partial z}R^{-1}(z, \mathbf{t}) = F_{\mathcal{P}, \alpha}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_{\mathcal{ZP}, \alpha}^{-1}(\mathbf{t})(zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1}G_{\mathcal{Z}, \alpha}(\mathbf{t}).$$

The representations (14.18), (14.19) above are local: each of them holds in the appropriate individual subset \mathcal{D}_α . Note, however, that whenever $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''} \neq \emptyset$, by Theorem 14.5 we have

$$S_{\mathcal{ZP}, \alpha''}(\mathbf{t}) = D_{\mathcal{Z}, \alpha', \alpha''}^{-1}(\mathbf{t})S_{\mathcal{ZP}, \alpha'}(\mathbf{t})D_{\mathcal{P}, \alpha', \alpha''}(\mathbf{t}), \quad \mathbf{t} \in \mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''},$$

where the functions $D_{\mathcal{Z}, \alpha', \alpha''}(\mathbf{t})$, $D_{\mathcal{P}, \alpha', \alpha''}(\mathbf{t})$ are as in (14.12). Hence the expressions (14.18), (14.19) coincide in the intersections of the subsets \mathcal{D}_α (although the individual functions $F_{\mathcal{P}, \alpha}(\mathbf{t})$, $S_{\mathcal{ZP}, \alpha}(\mathbf{t})$, $G_{\mathcal{Z}, \alpha}(\mathbf{t})$ do not). Here it is a self-evident fact, because these expressions represent globally defined objects. In the next section, where we shall use such local representations to *construct* globally defined objects, it will become a requirement.

15. HOLOMORPHIC FAMILIES OF GENERIC RATIONAL MATRIX FUNCTIONS WITH PRESCRIBED LOCAL DATA

This section can be considered as a \mathbf{t} -dependent version of Section 13. Here we consider the problem, how to construct a normalized holomorphic family of generic rational functions¹² with prescribed local data. The nature of such data is suggested by the considerations of the previous section (see, in particular, Theorem 14.5).

Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{\mathcal{D}_\alpha\}_{\alpha \in \mathfrak{A}}$ be a finite open covering of \mathcal{D} . We assume that $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ and $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ are collections of, respectively, $\mathbb{C}^{m \times n}$ -valued and $\mathbb{C}^{n \times m}$ -valued functions, satisfying the following conditions:

- (1) For each $\alpha \in \mathfrak{A}$ the functions $F_\alpha(\mathbf{t})$, $G_\alpha(\mathbf{t})$ are holomorphic in \mathcal{D}_α .
- (2) Whenever $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''} \neq \emptyset$, there exist diagonal matrix functions $D_{r,\alpha',\alpha''}(\mathbf{t})$, $D_{c,\alpha',\alpha''}(\mathbf{t})$, holomorphic and invertible in $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$, such that for every $\mathbf{t} \in \mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$

$$(15.1) \quad F_{\alpha''}(\mathbf{t}) = F_{\alpha'}(\mathbf{t})D_{c,\alpha',\alpha''}(\mathbf{t}), \quad G_{\alpha''}(\mathbf{t}) = D_{r,\alpha',\alpha''}(\mathbf{t})G_{\alpha'}(\mathbf{t}).$$

Remark 15.1. Conditions 1. and 2. imply that the collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ and $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ represent holomorphic mappings from \mathcal{D} into the spaces¹³ $\mathbb{P}_c^{(m-1) \times n}$ and $\mathbb{P}_r^{n \times (m-1)}$, respectively.

For each $\alpha \in \mathfrak{A}$ and $\mathbf{t} \in \mathcal{D}_\alpha$ we consider the Lyapunov equation

$$(15.2) \quad A_{\mathcal{P}}(\mathbf{t})S_\alpha(\mathbf{t}) - S_\alpha(\mathbf{t})A_{\mathcal{Z}}(\mathbf{t}) = G_\alpha(\mathbf{t})F_\alpha(\mathbf{t}),$$

where $A_{\mathcal{P}}(\mathbf{t})$ and $A_{\mathcal{Z}}(\mathbf{t})$ are as in (14.1). Its solution $S_\alpha(\mathbf{t})$ is the \mathcal{PZ} -coupling¹⁴ matrix, related to the data $A_{\mathcal{P}}(\mathbf{t})$, $A_{\mathcal{Z}}(\mathbf{t})$, $F_\alpha(\mathbf{t})$, $G_\alpha(\mathbf{t})$. Considered as a function of \mathbf{t} , $S_\alpha(\mathbf{t})$ is holomorphic in \mathcal{D}_α , because the right-hand side $G_\alpha(\mathbf{t})F_\alpha(\mathbf{t})$ is holomorphic in \mathcal{D}_α . The collection of functions $\{S_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is said to be the collection of \mathcal{PZ} -coupling functions related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$. In view of (15.1), whenever $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''} \neq \emptyset$, we have

$$(15.3) \quad S_{\alpha''}(\mathbf{t}) = D_{r,\alpha',\alpha''}(\mathbf{t})S_{\alpha'}(\mathbf{t})D_{c,\alpha',\alpha''}(\mathbf{t}), \quad \mathbf{t} \in \mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''},$$

where $D_{r,\alpha',\alpha''}(\mathbf{t})$, $D_{c,\alpha',\alpha''}(\mathbf{t})$ are diagonal, holomorphic and invertible matrix functions. Hence either $\forall \alpha \in \mathfrak{A} \det S_\alpha(\mathbf{t}) \equiv 0$ or $\forall \alpha \in \mathfrak{A} \det S_\alpha(\mathbf{t}) \not\equiv 0$. In the latter case the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ and $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is said to be \mathcal{PZ} -admissible, and the set

$$(15.4) \quad \Gamma_{\mathcal{PZ}} = \bigcup_{\alpha \in \mathfrak{A}} \{\mathbf{t} \in \mathcal{D}_\alpha : \det S_\alpha(\mathbf{t}) = 0\}$$

is said to be the \mathcal{PZ} -singular set related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$.

In the same way, we can consider the collection of functions $\{S_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, where for each $\alpha \in \mathfrak{A}$ and $\mathbf{t} \in \mathcal{D}_\alpha$ the matrix $S_\alpha(\mathbf{t})$ is the solution of the Lyapunov equation

$$(15.5) \quad A_{\mathcal{Z}}(\mathbf{t})S_\alpha(\mathbf{t}) - S_\alpha(\mathbf{t})A_{\mathcal{P}}(\mathbf{t}) = G_\alpha(\mathbf{t})F_\alpha(\mathbf{t}).$$

This collection is said to be the collection of \mathcal{ZP} -coupling functions related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$. If for every $\alpha \in \mathfrak{A} \det S_\alpha(\mathbf{t}) \not\equiv 0$ in \mathcal{D}_α

¹²See Definition 14.1.

¹³See Definition 13.5.

¹⁴See Definition 13.3.

then the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is said to be \mathcal{ZP} -admissible, and the set

$$(15.6) \quad \Gamma_{\mathcal{ZP}} = \bigcup_{\alpha \in \mathfrak{A}} \{\mathbf{t} \in \mathcal{D}_\alpha : \det S_\alpha(\mathbf{t}) = 0\}$$

is said to be the \mathcal{ZP} -singular set related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$.

Remark 15.2. Note that, in view of (15.1), if for each $\alpha \in \mathfrak{A}$ $S_\alpha(\mathbf{t})$ satisfies the Lyapunov equation (15.2) (or (15.5)) then the subsets

$$\Gamma_\alpha = \{\mathbf{t} \in \mathcal{D}_\alpha : \det S_\alpha(\mathbf{t}) = 0\}$$

of the appropriate singular set agree in the intersections of the sets \mathcal{D}_α :

$$\Gamma_{\alpha'} \cap (\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}) = \Gamma_{\alpha''} \cap (\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}) \quad \forall \alpha', \alpha''.$$

Theorem 15.3. *Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{\mathcal{D}_\alpha\}_{\alpha \in \mathfrak{A}}$ be a finite open covering of \mathcal{D} . Let $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ and $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ are collections of, respectively, $\mathbb{C}^{m \times n}$ -valued and $\mathbb{C}^{n \times m}$ -valued functions, satisfying the following conditions:*

- (1) *For each $\alpha \in \mathfrak{A}$ the functions $F_\alpha(\mathbf{t})$, $G_\alpha(\mathbf{t})$ are holomorphic in \mathcal{D}_α .*
- (2) *Whenever $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''} \neq \emptyset$, there exist diagonal matrix functions $D_{r,\alpha',\alpha''}(\mathbf{t})$, $D_{c,\alpha',\alpha''}(\mathbf{t})$, holomorphic and invertible in $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$, such that for every $\mathbf{t} \in \mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$*

$$F_{\alpha''}(\mathbf{t}) = F_{\alpha'}(\mathbf{t})D_{c,\alpha',\alpha''}(\mathbf{t}), \quad G_{\alpha''}(\mathbf{t}) = D_{r,\alpha',\alpha''}(\mathbf{t})G_{\alpha'}(\mathbf{t}).$$

- (3) *The pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is \mathcal{ZP} -admissible.*

Let $\Gamma_{\mathcal{ZP}}$ denote the \mathcal{ZP} -singular set related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$. Then there exists a unique normalized holomorphic family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D} \setminus \Gamma_{\mathcal{ZP}}}$ of rational generic functions parameterized by the pole and zero loci such that for every $\alpha \in \mathfrak{A}$ and $\mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}}$ the matrices $F_\alpha(\mathbf{t})$ and $G_\alpha(\mathbf{t})$ are, respectively, the left pole and right zero semiresidual matrices of $R(\cdot, \mathbf{t})$:

$$(15.7) \quad F_\alpha(\mathbf{t}) = F_{\mathcal{P},\alpha}(\mathbf{t}), \quad G_\alpha(\mathbf{t}) = G_{\mathcal{Z},\alpha}(\mathbf{t}), \quad \forall \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}}.$$

It is locally given by

$$(15.8) \quad R(z, \mathbf{t}) = I - F_\alpha(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_\alpha^{-1}(\mathbf{t})G_\alpha(\mathbf{t}), \quad \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}},$$

where $\{S_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is the collection of \mathcal{ZP} -coupling functions related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$. Furthermore, the logarithmic derivative of $R(z, \mathbf{t})$ with respect to z admits the local representation

$$(15.9) \quad \frac{\partial R(z, \mathbf{t})}{\partial z} R^{-1}(z, \mathbf{t}) = \\ = F_\alpha(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_\alpha^{-1}(\mathbf{t})(zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1}G_\alpha(\mathbf{t}), \\ \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}}.$$

Proof. In view of Proposition 13.4 and condition 3, for each $\alpha \in \mathfrak{A}$ and $\mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}}$ there exists a unique generic rational function $R_\alpha(\cdot, \mathbf{t})$, normalized by $R_\alpha(\infty, \mathbf{t}) = I$, with the pole and zero matrices $A_{\mathcal{P}}(\mathbf{t})$, $A_{\mathcal{Z}}(\mathbf{t})$ and the prescribed left zero and right pole semiresidual matrices (15.7). The function $R_\alpha(\cdot, \mathbf{t})$ and its logarithmic derivative admit the representations

$$(15.10) \quad R_\alpha(z, \mathbf{t}) = I - F_\alpha(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_\alpha^{-1}(\mathbf{t})G_\alpha(\mathbf{t}),$$

$$\begin{aligned}
R'_\alpha(z, \mathbf{t}) R_\alpha^{-1}(z, \mathbf{t}) &= \\
&= F_\alpha(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1} S_\alpha^{-1}(\mathbf{t})(zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1} G_\alpha(\mathbf{t}).
\end{aligned}$$

From the representation (15.10) and condition 1 it follows that the family $\{R_\alpha(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}}}$ is holomorphic. In view of condition 2 (see also (15.3)), we have

$$R_{\alpha'}(z, \mathbf{t}) = R_{\alpha''}(z, \mathbf{t}), \quad \forall \mathbf{t} \in (\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}) \setminus \Gamma_{\mathcal{ZP}}.$$

Hence we can define the holomorphic family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D} \setminus \Gamma_{\mathcal{ZP}}}$ by

$$R(z, \mathbf{t}) = R_\alpha(z, \mathbf{t}), \quad \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{ZP}}$$

to obtain the local representations (15.8), (15.9). The uniqueness of such a family follows from the uniqueness of each function $R_\alpha(\cdot, \mathbf{t})$. \square

Theorem 15.4. *Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{\mathcal{D}_\alpha\}_{\alpha \in \mathfrak{A}}$ be a finite open covering of \mathcal{D} . Let $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ and $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ are collections of, respectively, $\mathbb{C}^{m \times n}$ -valued and $\mathbb{C}^{n \times m}$ -valued functions, satisfying the following conditions:*

- (1) *For each $\alpha \in \mathfrak{A}$ the functions $F_\alpha(\mathbf{t})$, $G_\alpha(\mathbf{t})$ are holomorphic in \mathcal{D}_α .*
- (2) *Whenever $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''} \neq \emptyset$, there exist diagonal matrix functions $D_{r, \alpha', \alpha''}(\mathbf{t})$, $D_{c, \alpha', \alpha''}(\mathbf{t})$, holomorphic and invertible in $\mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$, such that for every $\mathbf{t} \in \mathcal{D}_{\alpha'} \cap \mathcal{D}_{\alpha''}$*

$$F_{\alpha''}(\mathbf{t}) = F_{\alpha'}(\mathbf{t}) D_{c, \alpha', \alpha''}(\mathbf{t}), \quad G_{\alpha''}(\mathbf{t}) = D_{r, \alpha', \alpha''}(\mathbf{t}) G_{\alpha'}(\mathbf{t}).$$

- (3) *The pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is \mathcal{PZ} -admissible.*

Let $\Gamma_{\mathcal{PZ}}$ denote the \mathcal{PZ} -singular set related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$. Then there exists a unique normalized holomorphic family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D} \setminus \Gamma_{\mathcal{PZ}}}$ of rational generic functions parameterized by the pole and zero loci such that for every $\alpha \in \mathfrak{A}$ and $\mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{PZ}}$ the matrices $F_\alpha(\mathbf{t})$ and $G_\alpha(\mathbf{t})$ are, respectively, the left zero and right pole semiresidual matrices of $R(\cdot, \mathbf{t})$:

$$(15.11) \quad F_\alpha(\mathbf{t}) = F_{\mathcal{Z}, \alpha}(\mathbf{t}), \quad G_\alpha(\mathbf{t}) = G_{\mathcal{P}, \alpha}(\mathbf{t}), \quad \forall \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{PZ}}.$$

It is locally given by

$$(15.12) \quad R(z, \mathbf{t}) = I + F_\alpha(\mathbf{t}) S_\alpha^{-1}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1} G_\alpha(\mathbf{t}), \quad \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{PZ}},$$

where $\{S_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$ is the collection of \mathcal{PZ} -coupling functions related to the pair of collections $\{F_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$, $\{G_\alpha(\mathbf{t})\}_{\alpha \in \mathfrak{A}}$. Furthermore, the logarithmic derivative of $R(\cdot, \mathbf{t})$ admits the local representation

$$\begin{aligned}
(15.13) \quad \frac{\partial R(z, \mathbf{t})}{\partial z} R^{-1}(z, \mathbf{t}) &= \\
&= -F_\alpha(\mathbf{t}) S_\alpha^{-1}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1} S_\alpha(\mathbf{t})(zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1} S_\alpha^{-1}(\mathbf{t}) G_\alpha(\mathbf{t}), \\
&\quad \mathbf{t} \in \mathcal{D}_\alpha \setminus \Gamma_{\mathcal{PZ}}.
\end{aligned}$$

Proof. The proof is analogous to that of Theorem 15.3. \square

16. ISOSEMIRESIDUAL FAMILIES OF GENERIC RATIONAL MATRIX FUNCTIONS

In the present section we shall consider an important special case of holomorphic families of generic rational functions, parameterized by the pole and zero loci \mathbf{t} . Namely, we are interested in the case when (either left pole and right zero or right pole and left zero) semiresidual functions of \mathbf{t} , determining the family as explained in Section 14, are constant.

Definition 16.1. Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions, parameterized by the pole and zero loci.

- (1) The family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ is said to be \mathcal{ZP} -*isosemiresidual*¹⁵ if there exists a pair of matrices $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ such that for every $\mathbf{t} \in \mathcal{D}$ the matrices F and G are, respectively, the left pole and right zero semiresidual matrices of the generic rational function $R(\cdot, \mathbf{t})$:

$$F = F_{\mathcal{P}}(\mathbf{t}), \quad G = G_{\mathcal{Z}}(\mathbf{t}), \quad \forall \mathbf{t} \in \mathcal{D}.$$

- (2) The family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ is said to be \mathcal{PZ} -*isosemiresidual* if there exists a pair of matrices $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ such that for every $\mathbf{t} \in \mathcal{D}$ the matrices F and G are, respectively, the left zero and right pole semiresidual matrices of the generic rational function $R(\cdot, \mathbf{t})$:

$$F = F_{\mathcal{Z}}(\mathbf{t}), \quad G = G_{\mathcal{P}}(\mathbf{t}), \quad \forall \mathbf{t} \in \mathcal{D}.$$

Let us assume that a pair of matrices $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ is given. How to construct a (\mathcal{PZ} - or \mathcal{ZP} -) isosemiresidual normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions, parameterized by the pole and zero loci, for which the constant functions

$$(16.1) \quad F(\mathbf{t}) \equiv F, \quad G(\mathbf{t}) \equiv G$$

would be the appropriate semiresidual functions? This is a special case of the problem considered in Section 15 (see Theorem 15.3). Note, however, that in this case the prescribed semiresidual functions (16.1) are holomorphic in the domain \mathbb{C}_*^{2n} . Therefore, we may consider its open covering consisting of the single set – the domain itself.

Following the approach described in Section 15, we consider the solutions $S_{\mathcal{PZ}}(\mathbf{t})$, $S_{\mathcal{ZP}}(\mathbf{t})$ of the Lyapunov equations

$$(16.2a) \quad A_{\mathcal{P}}(\mathbf{t})S_{\mathcal{PZ}}(\mathbf{t}) - S_{\mathcal{PZ}}(\mathbf{t})A_{\mathcal{Z}}(\mathbf{t}) = GF,$$

$$(16.2b) \quad A_{\mathcal{Z}}(\mathbf{t})S_{\mathcal{ZP}}(\mathbf{t}) - S_{\mathcal{ZP}}(\mathbf{t})A_{\mathcal{P}}(\mathbf{t}) = GF,$$

where

$$(16.3) \quad A_{\mathcal{P}}(\mathbf{t}) = \text{diag}(t_1, \dots, t_n), \quad A_{\mathcal{Z}}(\mathbf{t}) = \text{diag}(t_{n+1}, \dots, t_{2n}).$$

Then the functions $S_{\mathcal{PZ}}(\mathbf{t})$, $S_{\mathcal{ZP}}(\mathbf{t})$ are given explicitly by

$$(16.4a) \quad S_{\mathcal{PZ}}(\mathbf{t}) = \left(\frac{g_i f_j}{t_i - t_{n+j}} \right)_{1 \leq i, j \leq n},$$

$$(16.4b) \quad S_{\mathcal{ZP}}(\mathbf{t}) = \left(\frac{g_i f_j}{t_{n+i} - t_j} \right)_{1 \leq i, j \leq n},$$

where g_i and f_j denote, respectively, the i -th row of G and the j -th column of F . In particular, the functions $S_{\mathcal{PZ}}(\mathbf{t})$, $S_{\mathcal{ZP}}(\mathbf{t})$ are rational with respect to \mathbf{t} and holomorphic in \mathbb{C}_*^{2n} . The next step is to verify that the constant functions (16.1) are \mathcal{PZ} - or \mathcal{ZP} -admissible (that is, suitable for the construction of a holomorphic family of generic rational functions – see Theorem 15.3). This means to check that $\det S_{\mathcal{PZ}}(\mathbf{t}) \neq 0$ or $\det S_{\mathcal{ZP}}(\mathbf{t}) \neq 0$. Note that, since the functions $S_{\mathcal{PZ}}(\mathbf{t})$, $S_{\mathcal{ZP}}(\mathbf{t})$ are identical up to the permutation of variables $t_k \leftrightarrow t_{n+k}$, $1 \leq k \leq n$, these conditions are equivalent.

¹⁵Iso- (from $\iota\sigma\sigma\varsigma$ - equal- in Old Greek) is a combining form.

Definition 16.2. A pair of matrices $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ is said to be *admissible* if the $\mathbb{C}^{n \times n}$ -valued rational functions $S_{\mathcal{PZ}}(\mathbf{t})$, $S_{\mathcal{ZP}}(\mathbf{t})$ given by (16.4) satisfy the (equivalent) conditions

$$\det S_{\mathcal{PZ}}(\mathbf{t}) \neq 0, \quad \det S_{\mathcal{ZP}}(\mathbf{t}) \neq 0.$$

It turns out that the admissibility of a given pair of matrices $F \in \mathbb{C}^{m \times n}$, $G \in \mathbb{C}^{n \times m}$ can be checked by means of a simple criterion, described below.

Recall that for a matrix $M = (m_{i,j})_{i,j=1}^n$,

$$(16.5) \quad \det M = \sum_{\sigma} (-1)^{\sigma} m_{1,\sigma(1)} \cdots m_{n,\sigma(n)},$$

where σ runs over all $n!$ permutations of the set $1, \dots, n$, and $(-1)^{\sigma}$ is equal to either 1 or -1 depending on the parity of the permutation σ .

Definition 16.3. A matrix $M \in \mathbb{C}^{n \times n}$ is said to be *Frobenius-singular* if for some ℓ , $1 \leq \ell \leq n$, there exist indices $1 \leq \alpha_1 < \cdots < \alpha_{\ell} \leq n$; $1 \leq \beta_1 < \cdots < \beta_{n-\ell+1} \leq n$, such that $m_{\alpha_i, \beta_j} = 0$ for all $1 \leq i \leq \ell$, $1 \leq j \leq n - \ell + 1$.

Theorem 16.4. A matrix $M \in \mathbb{C}^{n \times n}$ is Frobenius-singular if and only if all $n!$ summands $(-1)^{\sigma} m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}$ of the sum (16.5) representing the determinant $\det M$ are equal to zero.

Theorem 16.4 is due to G.Frobenius, [Fro1]. The proof of this theorem can be also found in [Ber], Chapter 10, Theorem 9. The book [LoPl] contains some historical remarks concerning this theorem. See the Preface of [LoPl], especially pp. xiii - xvii of the English edition (to which pp. 14 - 18 of the Russian translation correspond).

Proposition 16.5. A pair of matrices $F \in \mathbb{C}^{m \times n}$, $G \in \mathbb{C}^{n \times m}$ is admissible if and only if their product GF is not a Frobenius-singular matrix.

Proof. Assume first that the matrix GF is Frobenius-singular. Then, according to (16.4a),

$$(16.6) \quad \det S_{\mathcal{PZ}}(\mathbf{t}) = \sum_{\sigma} (-1)^{\sigma} \frac{m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}}{(t_1 - t_{1+\sigma(1)}) \cdots (t_n - t_{n+\sigma(n)})},$$

where σ runs over all $n!$ permutations of the set $1, \dots, n$ and $m_{i,j} = g_i f_{n+j}$. If the matrix GF is Frobenius-singular then, according to Theorem 16.4, all the numerators $m_{1,\sigma(1)} \cdots m_{n,\sigma(n)}$ of the summands in (16.6) are equal to zero, and hence $\det S_{\mathcal{PZ}}(\mathbf{t}) \equiv 0$.

Conversely, if the matrix GF is not Frobenius-singular then, according to the same Theorem 16.4, there exists a permutation σ_0 such that

$$m_{1,\sigma_0(1)} \cdots m_{n,\sigma_0(n)} \neq 0.$$

Let us choose and fix n pairwise different numbers t_1^0, \dots, t_n^0 and set $t_{n+\sigma_0(1)}^0 = t_1^0 - \varepsilon, \dots, t_{n+\sigma_0(n)}^0 = t_n^0 - \varepsilon$, where $\varepsilon \neq 0$. Then as $\varepsilon \rightarrow 0$ the summand

$$(-1)^{\sigma_0} \frac{m_{1,\sigma_0(1)} \cdots m_{n,\sigma_0(n)}}{(t_1 - t_{n+\sigma_0(1)}(\varepsilon)) \cdots (t_n - t_{n+\sigma_0(n)}(\varepsilon))} = (-1)^{\sigma_0} (m_{1,\sigma_0(1)} \cdots m_{n,\sigma_0(n)}) \varepsilon^{-n}$$

is the leading term of the sum on the right-hand side of (16.6): all other summands grow at most as $O(\varepsilon^{-(n-1)})$. \square

Definition 16.6. Let a pair of matrices $F \in \mathbb{C}^{m \times n}$, $G \in \mathbb{C}^{n \times m}$ be such that their product GF is *not* a Frobenius-singular matrix and let the $\mathbb{C}^{n \times n}$ -valued rational functions $S_{\mathcal{P}\mathcal{Z}}(\mathbf{t})$, $S_{\mathcal{Z}\mathcal{P}}(\mathbf{t})$ be given by (16.4).

(1) The set

$$(16.7) \quad \Gamma_{\mathcal{P}\mathcal{Z}} = \{\mathbf{t} \in \mathbb{C}_*^{2n} : \det S_{\mathcal{P}\mathcal{Z}}(\mathbf{t}) = 0\}$$

is said to be the $\mathcal{P}\mathcal{Z}$ -singular set related to the pair F, G .

(2) The set

$$(16.8) \quad \Gamma_{\mathcal{Z}\mathcal{P}} = \{\mathbf{t} \in \mathbb{C}_*^{2n} : \det S_{\mathcal{Z}\mathcal{P}}(\mathbf{t}) = 0\}$$

is said to be the $\mathcal{Z}\mathcal{P}$ -singular set related to the pair F, G .

Remark 16.7. Note that, since $\det S_{\mathcal{P}\mathcal{Z}}(\mathbf{t})$, $\det S_{\mathcal{Z}\mathcal{P}}(\mathbf{t})$ are polynomials in $(t_i - t_j)^{-1}$, the singular sets $\Gamma_{\mathcal{P}\mathcal{Z}}$, $\Gamma_{\mathcal{Z}\mathcal{P}}$ related to the pair F, G are complex algebraic varieties of codimension one in \mathbb{C}_*^{2n} .

Combining Proposition 16.5 and Theorems 15.3, 15.4, we obtain

Theorem 16.8. *Let matrices $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ be such that their product GF is not a Frobenius-singular matrix, and let $\Gamma_{\mathcal{P}\mathcal{Z}}$, $\Gamma_{\mathcal{Z}\mathcal{P}}$ be the related singular sets. Then the following statements hold true.*

(1) *There exists a unique $\mathcal{P}\mathcal{Z}$ -isosemiresidual family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{P}\mathcal{Z}}}$ of $\mathbb{C}^{m \times m}$ -valued generic rational functions such that for every $\mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{P}\mathcal{Z}}$ the matrices F and G are, respectively, the left zero and right pole semiresidual matrices of $R(\cdot, \mathbf{t})$:*

$$(16.9) \quad F = F_{\mathcal{Z}}(\mathbf{t}), \quad G = G_{\mathcal{P}}(\mathbf{t}), \quad \forall \mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{P}\mathcal{Z}}.$$

It is given by

$$(16.10) \quad R(z, \mathbf{t}) = I + FS_{\mathcal{P}\mathcal{Z}}^{-1}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}G, \quad \mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{P}\mathcal{Z}}$$

where the function $S_{\mathcal{P}\mathcal{Z}}(\mathbf{t})$ satisfying the equation (16.2a) is given by (16.4a). Furthermore, the logarithmic derivative of $R(z, \mathbf{t})$ with respect to z admits the representation

$$(16.11) \quad \begin{aligned} \frac{\partial R(z, \mathbf{t})}{\partial z} R^{-1}(z, \mathbf{t}) = \\ = -FS_{\mathcal{P}\mathcal{Z}}^{-1}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_{\mathcal{P}\mathcal{Z}}(\mathbf{t})(zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1}S_{\mathcal{P}\mathcal{Z}}^{-1}(\mathbf{t})G, \\ \mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{P}\mathcal{Z}}. \end{aligned}$$

(2) *There exists a unique $\mathcal{Z}\mathcal{P}$ -isosemiresidual family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{Z}\mathcal{P}}}$ of $\mathbb{C}^{m \times m}$ -valued generic rational functions such that for every $\mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{Z}\mathcal{P}}$ the matrices F and G are, respectively, the left pole and right zero semiresidual matrices of $R(\cdot, \mathbf{t})$:*

$$(16.12) \quad F = F_{\mathcal{P}}(\mathbf{t}), \quad G = G_{\mathcal{Z}}(\mathbf{t}), \quad \forall \mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{Z}\mathcal{P}}.$$

It is given by

$$(16.13) \quad R(z, \mathbf{t}) = I - F(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1}S_{\mathcal{Z}\mathcal{P}}^{-1}(\mathbf{t})G, \quad \mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{Z}\mathcal{P}},$$

where the function $S_{\mathcal{ZP}}(\mathbf{t})$ satisfying the equation (16.2b) is given by (16.4b). Furthermore, the logarithmic derivative of $R(z, \mathbf{t})$ with respect to z admits the representation

$$(16.14) \quad \frac{\partial R(z, \mathbf{t})}{\partial z} R^{-1}(z, \mathbf{t}) = F(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1} S_{\mathcal{ZP}}^{-1}(\mathbf{t}) (zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1} G, \\ \mathbf{t} \in \mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{ZP}}.$$

17. ISOPRINCIPAL FAMILIES OF GENERIC RATIONAL MATRIX FUNCTIONS

Our interest in holomorphic families of generic rational functions is motivated by our intent to construct rational solutions of the Schlesinger system (see Section 18 below). Indeed, given a holomorphic family $\{R(z, \mathbf{t}) : \mathbf{t} \in \mathcal{D}\}$ of generic rational functions, we can consider the linear differential system

$$(17.1) \quad \frac{\partial R(z, \mathbf{t})}{\partial z} = Q_R(z, \mathbf{t}) R(z, \mathbf{t}),$$

where $Q_R(z, \mathbf{t})$ is the logarithmic derivative of $R(z, \mathbf{t})$ with respect to z . According to Lemma 14.2 (and in view of (14.3), (14.4c)), the system (17.1) can be rewritten as

$$(17.2) \quad \frac{\partial R(z, \mathbf{t})}{\partial z} = \left(\sum_{k=1}^{2n} \frac{Q_k(\mathbf{t})}{z - t_k} \right) R(z, \mathbf{t}),$$

where the functions $Q_k(\mathbf{t})$ are holomorphic in \mathcal{D} . The system (17.2) can be viewed as a holomorphic family (=deformation) of Fuchsian systems parameterized by the singularities' loci. It was proved in [KaVo] (Theorem 8.2) that in the case when the deformation (17.2) is *isoprincipal* the functions $Q_k(\mathbf{t})$ satisfy the Schlesinger system.

Definition 17.1. Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions, parameterized by the pole and zero loci. Assume that for $1 \leq k \leq 2n$ there exist $\mathbb{C}^{m \times m}$ -valued functions $E_k(\cdot)$, holomorphic and invertible in \mathbb{C}_* , such that for every $\mathbf{t} \in \mathcal{D}$ the function E_k is the principal factor¹⁶ of the function $R(\cdot, \mathbf{t})$ at t_k : there exists a $\mathbb{C}^{m \times m}$ -valued function $H_k(\cdot, \mathbf{t})$, holomorphic and invertible in a neighborhood of t_k , such that

$$(17.3) \quad R(z, \mathbf{t}) = H_k(z, \mathbf{t}) E_k(z - t_k).$$

Then the family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ is said to be *isoprincipal*.

Theorem 17.2. Let \mathcal{D} be a domain in \mathbb{C}_*^{2n} and let $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ be a normalized holomorphic family of $\mathbb{C}^{m \times m}$ -valued generic rational functions, parameterized by the pole and zero loci. The family $\{R(z, \mathbf{t}) : \mathbf{t} \in \mathcal{D}\}$ is *isoprincipal* if and only if it is *PZ-isosemiresidual*.

Proof. First, assume that the family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ is *PZ-isosemiresidual*. Then, according to Definition 16.1, there exist $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ such that for every $\mathbf{t} \in \mathcal{D}$ the matrices F and G are, respectively, the left zero and right pole semiresidual matrices of the generic rational function $R(\cdot, \mathbf{t})$. Then, by Lemma

¹⁶See Definition 10.10.

10.12 (see also Remark 10.14), for $k = 1, \dots, 2n$ and independently of \mathbf{t} the principal factor $E_k(\zeta)$ of $R(\cdot, \mathbf{t})$ at t_k can be chosen in the form

$$(17.4) \quad E_k(\zeta) = \begin{cases} I + L_k - \zeta^{-1}L_k, & \text{if } 1 \leq k \leq n, \\ I - L_k + \zeta L_k, & \text{if } 1+n \leq k \leq 2n, \end{cases}$$

where

$$(17.5) \quad L_k = \begin{cases} -g_k^*(g_k g_k^*)^{-1} g_k, & \text{if } 1 \leq k \leq n, \\ f_k(f_k^* f_k)^{-1} f_k^*, & \text{if } 1+n \leq k \leq 2n, \end{cases}$$

g_k is the k -th row of G and f_k is the $(k-n)$ -th column of F . Hence, by Definition 17.1 the family $\{R(z, \mathbf{t}) : \mathbf{t} \in \mathcal{D}\}$ is isoprincipal.

Conversely, assume that the family $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathcal{D}}$ is isoprincipal. Then, according to Definition 17.1, for $1 \leq k \leq 2n$ there exist functions $\hat{E}_k(\cdot)$, holomorphic and invertible in \mathbb{C}_* , such that for every $\mathbf{t} \in \mathcal{D}$ the function \hat{E}_k is the principal factor of the function $R(\cdot, \mathbf{t})$ at t_k . Let $\mathbf{t}^0, \mathbf{t} \in \mathcal{D}$ be fixed and let $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$ be, respectively, the left zero and right pole semiresidual matrices of the generic rational function $R(\cdot, \mathbf{t}^0)$. Let us denote by g_k the k -th row of G and by f_k the $(k-n)$ -th column of F . Then, in view of Remark 10.11, the function \hat{E}_k is of the form

$$\hat{E}_k(\zeta) = M_k(\zeta) E_k(\zeta),$$

where $M_k(\cdot)$ is a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible in \mathbb{C} and $E_k(\cdot)$ is given by (17.4), (17.5). Hence for z in a neighborhood of t_k $R(z, \mathbf{t})$ admits the representation

$$R(z, \mathbf{t}) = H_k(z, \mathbf{t}) M_k(z - t_k) E_k(z - t_k),$$

where $H_k(\cdot, \mathbf{t})$ is a $\mathbb{C}^{m \times m}$ -valued function, holomorphic and invertible at t_k . Then, for $k = 1, \dots, n$, the residue $R_k(\mathbf{t})$ of $R(z, \mathbf{t})$ at t_k is given by

$$R_k(\mathbf{t}) = (H_k(t_k, \mathbf{t}) M_k(0) g_k^* (g_k g_k^*)^{-1}) g_k.$$

Therefore, G is the right pole semiresidual matrix of the function $R(z, \mathbf{t})$, as well. Analogously, for $k = n+1, \dots, 2n$

$$E^{-1}(\zeta) = I - L_k + \zeta^{-1} L_k,$$

where

$$L_k = f_k(f_k^* f_k)^{-1} f_k^*,$$

hence the residue $R_k(\mathbf{t})$ of $R^{-1}(z, \mathbf{t})$ at t_k is given by

$$R_k(\mathbf{t}) = f_k((f_k^* f_k)^{-1} f_k^* M_k^{-1}(0) H_k^{-1}(t_k, \mathbf{t})).$$

Therefore, F is the left zero semiresidual matrix of the function $R(z, \mathbf{t})$, as well. This completes the proof. \square

Theorem 17.2 reduces the construction of an *isoprincipal* family to the construction of an *isosemiresidual* family. The latter problem has already been considered in Section 16. According to Theorems 16.8 and 17.2, from any pair of matrices $F \in \mathbb{C}^{m \times n}$ and $G \in \mathbb{C}^{n \times m}$, such that the product GF is not a Frobenius-singular matrix, we can construct an isoprincipal family of generic rational functions $\{R(z, \mathbf{t})\}_{\mathbf{t} \in \mathbb{C}^{2n} \setminus \Gamma_{\mathcal{PZ}}}$, where $\Gamma_{\mathcal{PZ}}$ denotes the \mathcal{PZ} -singular set related to the pair of matrices F, G . This family is given by

$$(17.6) \quad R(z, \mathbf{t}) = I + F S_{\mathcal{PZ}}^{-1}(\mathbf{t})(zI - A_{\mathcal{P}}(\mathbf{t}))^{-1} G,$$

where the function $S_{\mathcal{PZ}}(\mathbf{t})$, satisfying (16.2a), is given by (16.4a). The logarithmic derivative of $R(z, \mathbf{t})$ with respect to z is given by

$$(17.7) \quad \frac{\partial R(z, \mathbf{t})}{\partial z} R^{-1}(z, \mathbf{t}) = -F S_{\mathcal{PZ}}^{-1}(\mathbf{t}) (zI - A_{\mathcal{P}}(\mathbf{t}))^{-1} S_{\mathcal{PZ}}(\mathbf{t}) (zI - A_{\mathcal{Z}}(\mathbf{t}))^{-1} S_{\mathcal{PZ}}^{-1}(\mathbf{t}) G,$$

and we obtain the following expressions for its residues $Q_k(\mathbf{t})$

$$(17.8a) \quad Q_k(\mathbf{t}) = -F S_{\mathcal{PZ}}^{-1}(\mathbf{t}) I_{[k]} S_{\mathcal{PZ}}(\mathbf{t}) (t_k I - A_{\mathcal{Z}}(\mathbf{t}))^{-1} S_{\mathcal{PZ}}^{-1}(\mathbf{t}) G, \quad 1 \leq k \leq n,$$

$$(17.8b) \quad Q_k(\mathbf{t}) = -F S_{\mathcal{PZ}}^{-1}(\mathbf{t}) (t_k I - A_{\mathcal{P}}(\mathbf{t}))^{-1} S_{\mathcal{PZ}}(\mathbf{t}) I_{[k-n]} S_{\mathcal{PZ}}^{-1}(\mathbf{t}) G, \quad n+1 \leq k \leq 2n.$$

Here we use the notation

$$I_{[k]} \stackrel{\text{def}}{=} \text{diag}(\delta_{1,k}, \dots, \delta_{n,k}),$$

where $\delta_{i,j}$ is the Kronecker delta.

Remark 17.3. Note that, according to (16.4a), the function $S_{\mathcal{PZ}}(\mathbf{t})$ is a rational function of \mathbf{t} . Hence also the functions $Q_k(\mathbf{t})$ are rational functions of \mathbf{t} .

18. RATIONAL SOLUTIONS OF THE SCHLESINGER SYSTEM

It can be checked that the rational functions $Q_k(\mathbf{t})$ given by (17.8) satisfy the Schlesinger system

$$(18.1) \quad \begin{cases} \frac{\partial Q_k}{\partial t_\ell} = \frac{[Q_\ell, Q_k]}{t_\ell - t_k}, & k \neq \ell, \\ \frac{\partial Q_k}{\partial t_k} = \sum_{\ell \neq k} \frac{[Q_\ell, Q_k]}{t_k - t_\ell}. \end{cases}$$

It is also not very difficult to check that

$$(18.2) \quad V(\mathbf{t}) \stackrel{\text{def}}{=} -F S_{\mathcal{PZ}}^{-1}(\mathbf{t}) G$$

is the potential function for this solution:

$$(18.3) \quad Q_k(\mathbf{t}) = \frac{\partial V(\mathbf{t})}{\partial t_k}, \quad k = 1, \dots, 2n.$$

Furthermore, one can show that the rational function $\det S_{\mathcal{PZ}}(\mathbf{t})$ admits the following integral representation

$$(18.4) \quad \det S_{\mathcal{PZ}}(\mathbf{t}) = \det S_{\mathcal{PZ}}(\mathbf{t}_0) \cdot \exp \left\{ \int_{\gamma} \sum_{\substack{1 \leq i, j \leq 2n, \\ j \neq i}} \frac{\text{trace} \left(\frac{\partial V(\mathbf{t})}{\partial t_i} \cdot \frac{\partial V(\mathbf{t})}{\partial t_j} \right)}{t_i - t_j} dt_i \right\}.$$

where \mathbf{t}_0 and \mathbf{t} are two arbitrary points the domain $\mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{PZ}}$, and γ is an arbitrary path which begins at \mathbf{t}_0 , ends at \mathbf{t} and is contained in $\mathbb{C}_*^{2n} \setminus \Gamma_{\mathcal{PZ}}$.

However, the explanation of these facts lies in the considerations of Sections 2 and 3 of the first part [KaVo] of this work. The matrix functions $Q_k(\mathbf{t})$ satisfy the

Schlesinger system, and the function $-V(\mathbf{t})$ is a Laurent coefficient at $z = \infty$ of the normalized solution (17.6) of the Fuchsian system

$$(18.5) \quad \frac{dR(z, \mathbf{t})}{dz} = \left(\sum_{1 \leq k \leq 2n} \frac{Q_k(\mathbf{t})}{z - t_k} \right) R(z, \mathbf{t}),$$

$$(18.6) \quad R(z, \mathbf{t}) = I - \frac{V(\mathbf{t})}{z} + o(|z|^{-1}) \quad \text{as } z \rightarrow \infty,$$

while the function

$$(18.7) \quad \tau(\mathbf{t}) \stackrel{\text{def}}{=} \det S_{\mathcal{P}\mathcal{Z}}(\mathbf{t}),$$

is the tau-function related to the solution $Q_1(\mathbf{t}), \dots, Q_{2n}(\mathbf{t})$ of the Schlesinger system.

More detailed explanation of these and other related facts will be given in the third part of this work.

APPENDIX

B. THE GLOBAL FACTORIZATION OF A HOLOMORPHIC MATRIX FUNCTION OF RANK ONE

Let $M(\mathbf{t}) = \|m_{p,q}(\mathbf{t})\|_{1 \leq p,q \leq m}$ be a $\mathbb{C}^{m \times m}$ -valued function of the variable $\mathbf{t} \in \mathcal{D}$, where \mathcal{D} is a domain in \mathbb{C}^N . (We can even assume that \mathcal{D} is a Riemann domain¹⁷ of dimension N over \mathbb{C}^N .) In our considerations $N = 2n$ and $\mathcal{D} \subseteq \mathbb{C}_*^{2n}$. Let the matrix function M be holomorphic in \mathcal{D} and let

$$(B.1) \quad \text{rank } M(\mathbf{t}) = 1 \quad \forall \mathbf{t} \in \mathcal{D}.$$

We will try to represent M in the form

$$(B.2) \quad M(\mathbf{t}) = f(\mathbf{t})g(\mathbf{t}),$$

where $f(\mathbf{t})$ and $g(\mathbf{t})$ are, respectively, a $\mathbb{C}^{m \times 1}$ -valued function and a $\mathbb{C}^{1 \times m}$ -valued function, both of them holomorphic¹⁸ in \mathcal{D} .

Let us recall that, according to Lemma 14.4, there exist a finite open covering $\{\mathcal{U}_p\}_{p=1}^m$ of \mathcal{D} , a collection $\{f_p(\mathbf{t})\}_{p=1}^m$ of $\mathbb{C}^{m \times 1}$ -valued functions and a collection $\{g_p(\mathbf{t})\}_{p=1}^m$ of $\mathbb{C}^{1 \times m}$ -valued functions satisfying the following conditions.

- (1) For $p = 1, \dots, m$ the functions $f_p(\mathbf{t})$ and $g_p(\mathbf{t})$ are holomorphic in \mathcal{U}_p .
- (2) For $p = 1, \dots, m$ the function $M(\mathbf{t})$ admits the factorization

$$(B.3) \quad M(\mathbf{t}) = f_p(\mathbf{t})g_p(\mathbf{t}), \quad \mathbf{t} \in \mathcal{U}_p.$$

- (3) Whenever $\mathcal{U}_{p'} \cap \mathcal{U}_{p''} \neq \emptyset$, there exists a (scalar) function $\varphi_{p',p''}(\mathbf{t})$, holomorphic and invertible in $\mathcal{U}_{p'} \cap \mathcal{U}_{p''}$, such that

$$(B.4) \quad f_{p''}(\mathbf{t}) = f_{p'}(\mathbf{t})\varphi_{p',p''}(\mathbf{t}), \quad g_{p''}(\mathbf{t}) = \varphi_{p',p''}^{-1}(\mathbf{t})g_{p'}(\mathbf{t}) \quad \forall \mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''}.$$

¹⁷See Definition 5.4.4 in [Hör].

¹⁸In general, such a global factorization is impossible even if the factors $f(\mathbf{t})$ and $g(\mathbf{t})$ are only required to be continuous rather than holomorphic: one of the obstacles is of topological nature.

In particular,

$$(B.5a) \quad \varphi_{p,p}(\mathbf{t}) = 1 \quad \forall \mathbf{t} \in \mathcal{U}_p,$$

$$(B.5b) \quad \varphi_{p,p'}(\mathbf{t}) = \varphi_{p,p''}(\mathbf{t})\varphi_{p'',p'}(\mathbf{t}) \quad \forall \mathbf{t} \in \mathcal{U}_p \cap \mathcal{U}_{p'} \cap \mathcal{U}_{p''}.$$

The equalities (B.3), $p = 1, \dots, k$, are nothing more than the factorizations of the form (B.2), with holomorphic factors $f_p(\mathbf{t})$ and $g_p(\mathbf{t})$. However, the factorization (B.3) is only *local*: for each p the equality (B.3) holds in the open subset \mathcal{U}_p of the set \mathcal{D} . For different p' and p'' , the factors $f_{p'}, g_{p'}$ and $f_{p''}, g_{p''}$ may *not agree in the intersections* $\mathcal{U}_{p'} \cap \mathcal{U}_{p''}$. To glue the factorizations (B.3) for different p together, we seek scalar functions $\varphi_p(\mathbf{t})$ which are holomorphic in \mathcal{U}_p , do not vanish there and satisfy the condition

$$(B.6) \quad f_{p'}(\mathbf{t})\varphi_{p'}(\mathbf{t}) = f_{p''}(\mathbf{t})\varphi_{p''}(\mathbf{t}) \quad \forall \mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''}.$$

Then, in view of (B.3),

$$(B.7) \quad \varphi_{p'}^{-1}(\mathbf{t})g_{p'}(\mathbf{t}) = \varphi_{p''}^{-1}(\mathbf{t})g_{p''}(\mathbf{t}) \quad \forall \mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''}.$$

Assuming that such functions φ_p , $1 \leq p \leq k$, are found, we set

$$(B.8a) \quad f(\mathbf{t}) \stackrel{\text{def}}{=} f_p(\mathbf{t})\varphi_p(\mathbf{t}) \text{ if } \mathbf{t} \in \mathcal{U}_p,$$

$$(B.8b) \quad g(\mathbf{t}) \stackrel{\text{def}}{=} \varphi_p^{-1}(\mathbf{t})g_p(\mathbf{t}) \text{ if } \mathbf{t} \in \mathcal{U}_p.$$

The relations (B.6), (B.7) ensure that these definitions are not contradictory. Thus the functions $f(\mathbf{t})$ and $g(\mathbf{t})$ are defined for every $\mathbf{t} \in \mathcal{D}$. Moreover, these functions are holomorphic in \mathcal{D} and provide the factorization (B.2).

From (B.4) it follows that the condition (B.6) is equivalent to the condition

$$(B.9) \quad \varphi_{p'}(\mathbf{t}) = \varphi_{p',p''}(\mathbf{t})\varphi_{p''}(\mathbf{t}) \quad \forall \mathbf{t} \in \mathcal{U}_{p'} \cap \mathcal{U}_{p''},$$

where $\varphi_{p',p''}(\mathbf{t})$ are the functions appearing in (B.4). Thus, to ensure that the conditions (B.6), (B.7) are in force, we have to solve the so-called *second Cousin problem* (see [Shab], [Oni], [Leit] and [Hör]):

Problem B.1. Let \mathcal{D} be a complex manifold and let $\{\mathcal{U}_\alpha\}_\alpha$ of \mathcal{D} be an open covering of \mathcal{D} . For each α, β such that $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ let a \mathbb{C} -valued function $\varphi_{\alpha,\beta}$, holomorphic and non-vanishing in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, be given.

Find a collection of \mathbb{C} -valued functions $\{\varphi_\alpha\}_\alpha$ with the following properties:

- (1) For every α the function φ_α is holomorphic in \mathcal{U}_α and does not vanish there.
- (2) Whenever $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, the relation

$$(B.10) \quad \varphi_\alpha = \varphi_{\alpha,\beta}\varphi_\beta$$

holds in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$.

A necessary condition for the solvability of the second Cousin problem in \mathcal{D} with the given data $\{\mathcal{U}_\alpha\}_\alpha$, $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta}$ is the so-called "*cocycle condition*":

$$(B.11) \quad \begin{aligned} \varphi_{\alpha,\gamma} &= \varphi_{\alpha,\beta}\varphi_{\beta,\gamma} && \text{in every non-empty triple intersection } \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma, \\ \varphi_{\alpha,\alpha} &= 1 && \text{in every } \mathcal{U}_\alpha. \end{aligned}$$

In our case this condition is fulfilled – see (B.5). However, the cocycle condition alone is not sufficient to guarantee the existence of a solution to the second Cousin problem – it depends on \mathcal{D} itself, as well.

Proposition B.2. (J.-P.Serre, [Ser1]; see also [Shab], section 16; [Hör], sections 5.5 and 7.4; [Oni], section 4.4.) *If \mathcal{D} is a Stein manifold¹⁹ which satisfies the condition²⁰*

$$(B.12) \quad H^2(\mathcal{D}, \mathbb{Z}) = 0,$$

then the second Cousin problem in \mathcal{D} with arbitrary given data $\{U_\alpha, \varphi_{\beta,\alpha}\}$ satisfying the cocycle condition (B.11) is solvable.

As we have seen, the factorization problem (B.2) can be reduced to solving the second Cousin with a certain data. Thus, the following result holds:

Theorem B.3. *Let $M(\mathbf{t})$ be a $\mathbb{C}^{m \times m}$ -valued function, holomorphic for $\mathbf{t} \in \mathcal{D}$, where \mathcal{D} is a Riemann domain over \mathbb{C}^N . Assume that M satisfies the condition*

$$\text{rank } M(\mathbf{t}) = 1 \quad \forall \mathbf{t} \in \mathcal{D}.$$

If \mathcal{D} possesses the property: the second Cousin problem in \mathcal{D} with arbitrary given data $\{U_\alpha\}_\alpha, \{\varphi_{\alpha,\beta}\}_{\alpha,\beta}$ satisfying the cocycle condition (B.11) is solvable, then the matrix function $M(\mathbf{t})$ admits the factorization of the form

$$M(\mathbf{t}) = f(\mathbf{t}) \cdot g(\mathbf{t}),$$

where the factors $f(\mathbf{t})$ and $g(\mathbf{t})$ are, respectively, a $\mathbb{C}^{m \times 1}$ -valued function and a $\mathbb{C}^{1 \times m}$ -valued function, holomorphic and non-vanishing for $\mathbf{t} \in \mathcal{D}$. In particular, such is the case if \mathcal{D} is a Stein manifold satisfying the condition (B.12).

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¹⁹A domain \mathcal{D} in \mathbb{C}^N is a Stein manifold if and only if \mathcal{D} is pseudoconvex, or, what is equivalent, \mathcal{D} is a holomorphy domain.

²⁰ $H^2(\mathcal{D}, \mathbb{Z})$ is the second cohomology group of \mathcal{D} with integer coefficients.

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